Introduction to differential algebraic geometry and differential algebraic groups
November 4, 18, December 2, 9, 2005

In his 1979 article on nonlinear differential equations, Manin describes three possible languages for the variational formalism: the classical language, the geometric language, and, differential algebra. He critiques each approach. He writes:

> The language of differential algebra is better suited for expressing such properties (invariant properties of differential equations), and, puts at the disposal of the investigator the extensive apparatus of commutative algebra, differential algebra, and algebraic geometry....The numerous "explicit formulas" for the solutions of the classical and newest differential equations have good interpretations in this language; the same may be said for conservation laws. However, the language of differential algebra which has been traditional since the work of Ritt does not contain the means for describing changes of the functions (dependent variables) and the variables x_i (independent variables), and for clarifying properties which are invariant under such changes. This is one of the main reasons for the embryonic state of so-called "Bäcklund transformations" in which there has been a recent surge of interest.

The development of differential algebraic geometry, which began in the 1970's has begun to address these concerns. There is a lot left to do. For a modern approach, see Kovacic (2002 ff)

1 Differential Algebraic Geometry

Throughout, $\mathcal{F}$ is a differential field of characteristic 0, equipped with a set ∂ of commuting derivation operators $\partial_1, \dots, \partial_m$, and, field $\mathcal{C}$ of constants. Note that the field $\mathbb{Q}$ of rational numbers is contained in $\mathcal{C}$. Let Θ be the free commutative monoid on ∂. If $\theta = \partial_1^{k_1} \cdots \partial_m^{k_m}$ is a derivative operator in Θ, then, $ord\theta = \sum k_i$. If the cardinality of ∂ is 1, we identify the set with its only element, and, call $\mathcal{F}$ an *ordinary differential field*. Otherwise, $\mathcal{F}$ is a *partial differential field*. Throughout, we will use the prefix ∂- in place of "differential" or "differentially."

Note that we define the ∂-structure on an $\mathcal{F}$-algebra $\mathcal{R}$ by a homomorphism from Θ into the multiplicative monoid $(End\mathcal{R},\cdot)$ that maps ∂ into $Der\mathcal{R}$. If $\mathcal{S}$ is a ∂-ring and a subring of a ∂-ring $\mathcal{R}$, then, the ∂-structure of $\mathcal{R}$ extends that of $\mathcal{S}$. In particular, the action of ∂ on $\mathcal{R}$ extends the action of ∂ on $\mathcal{F}$. We call $\mathcal{R}$ a ∂-$\mathcal{F}$-algebra. All our ∂-rings will be ∂-$\mathcal{F}$-algebras.

If $\mathcal{R}$ is a ∂-$\mathcal{F}$-algebra, $a = (a_1, \ldots, a_n)$ is a finite family of elements of $\mathcal{R}$, and $\theta \in \Theta$, we denote by θa the family $(\theta a_1, \ldots, \theta a_n)$, and, by Θa the family $\theta a, \theta \in \Theta$. $\mathcal{R}$ is *finitely ∂-generated over* $\mathcal{F}$ if there exists a finite family a of elements of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{F}[\Theta a]$. We write $\mathcal{R} = \mathcal{F}\{a\} = \mathcal{F}\{a_1, \ldots, a_n\}$. If $y = (y_1, \ldots, y_n)$ is a family of ∂-indeterminates, then, the ∂-polynomial algebra $\mathcal{F}\{y\} = \mathcal{F}\{y_1, \ldots, y_n\}$ over $\mathcal{F}$ is, thus, an infinitely generated polynomial algebra over $\mathcal{F}$. We can think of the ∂-polynomials as functions on $\mathcal{F}^n$.

If $\mathcal{G}$ is a ∂-extension field of $\mathcal{F}$, $\mathcal{G}$ is *finitely ∂-generated over* $\mathcal{F}$ if there exists a finite family a of elements of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{F}(\Theta a)$. We write $\mathcal{G} = \mathcal{F}\langle a \rangle$. $\mathcal{G}$ is the quotient field of $\mathcal{R} = \mathcal{F}\{a\}$.

Example 1.1. Let D be a connected open region of $\mathbb{C}^m$, $\mathbb{C}$ the field of complex numbers, and, let $t_1, \ldots, t_m$ be complex variables. Set $\partial = \{\partial_{t_1}, \ldots, \partial_{t_m}\}$. Let $\mathcal{G} = (\mathcal{G}, \partial)$ be the ∂-field of functions meromorphic in D. Let $\mathcal{F} = (\mathcal{F}, \partial)$ be the ∂-field of functions meromorphic in $\mathbb{C}^m$. Then, $\mathcal{G}$ is a ∂-extension field of $\mathcal{F}$.

Let $\mathcal{R}$, and, $\mathcal{S}$ be ∂-algebras over $\mathcal{F}$. An $\mathcal{F}$-algebra homomorphism $\varphi : \mathcal{R} \to \mathcal{S}$ is a *∂-homomorphism* if $\varphi \circ \partial_i = \partial_i \circ \varphi$ (φ commutes with the action of the derivation operators.) $\ker \varphi$ is a ∂-ideal of $\mathcal{R}$, and $\operatorname{im} \varphi$ is a ∂-subalgebra over $\mathcal{F}$ of $\mathcal{S}$.

Theorem 1.2. ***The Seidenberg Lefschetz Principle*** *(1958). Let $\mathcal{F}$ be any ∂-field that is finitely ∂-generated over $\mathbb{Q}$, where $\partial = \{\partial_1, \ldots, \partial_m\}$. Let $(\mathcal{M}, \partial)$ be the field of meromorphic functions on $\mathbb{C}^m$, where the action of ∂ is by the partial derivatives of the complex variables $t_1, \ldots, t_m$. Then, there is a connected open region D of $\mathbb{C}^m$ and, a ∂-isomorphism of $\mathcal{F}$ into the field of meromorphic functions on D.*

The logician Abraham Robinson formulated the analogue for differential algebra of an algebraically closed field (1959). Let $f_1, \ldots, f_r, g$ be differential polynomials of positive degree in $\mathcal{F}\{y_1, \ldots, y_n\}$. The system

$$f_1 = 0, \ldots, f_r = 0, g \neq 0$$

is *consistent* if there exists a ∂-extension field $\mathcal{G}$ of $\mathcal{F}$ and a family $a = (a_1, \ldots, a_n)$ of elements of $\mathcal{G}$ such that $f_1(a) = 0, \ldots, f_r(a) = 0, g(a) \neq 0$. $\mathcal{F}$ is *differentially closed* if every consistent system of differential polynomial equations and inequations has a solution with coordinates in $\mathcal{F}$. A differentially closed differential field is algebraically closed.

1.1 Differential affine n-space $\mathbb{A}^n$: The Kolchin topology

Let $\mathcal{U}$ be a differentially closed ∂-extension field of $\mathcal{F}$. Let $\mathcal{K}$ be the field of constants of $\mathcal{U}$. Recall the Zariski topology on $\mathcal{U}^n$: $V \subset \mathcal{U}^n$ is *Zariski closed* if there exist polynomials $(f_i)_{i\in I}$, $f_i \in \mathcal{U}[y_1, \ldots y_n]$ such that $V = \{a \in \mathcal{U}^n : f_i(\eta) = 0,\ i \in I\}$. The Zariski closed sets are the closed sets of the topology.whose closed sets are algebraic varieties. Now assume that $y_1, \ldots, y_n$ are ∂-indeterminates over $\mathcal{U}$.

Using the Zariski topology as a model, we define the *Kolchin topology* on $\mathcal{U}^n$. A subset V of $\mathcal{U}^n$ is *Kolchin closed* if there exist ∂-polynomials $(f_i)_{i\in I}$, $f_i \in \mathcal{U}\{y_1, \ldots y_n\}$ such that $V = \{a \in \mathcal{U}^n : f_i(a) = 0, i \in I\}$. The closed sets in the Kolchin topology are also called ∂-varieties. If the differential polynomials defining V have coefficients in $\mathcal{F}$, we say V is *defined over* $\mathcal{F}$, and call it a ∂-$\mathcal{F}$-variety.

If V is the set of zeros of $f_i, i \in I$, then, V is the set of zeros of the ∂-ideal $\mathfrak{i} = [(f_i)_{i\in I}]$ that they generate. We denote V by $V(\mathfrak{i})$. Conversely, if $\mathfrak{i}$ is a ∂-ideal of $\mathcal{U}\{y_1, \ldots y_n\}$, $V = V(\mathfrak{i})$ is Kolchin closed.

Note that V is order reversing:

$$\mathfrak{i} \subset \mathfrak{j} \Rightarrow V(\mathfrak{i}) \supset V(\mathfrak{j})$$

Also,

$$\begin{aligned} V([1]) &= \emptyset \\ V([0]) &= \mathcal{U}^n \\ V(\mathfrak{i} \cap \mathfrak{j}) &= V(\mathfrak{i}\mathfrak{j}) = V(\mathfrak{i}) \cup V(\mathfrak{j}) \\ V(\mathfrak{i} + \mathfrak{j}) &= V(\mathfrak{i}) \cap V(\mathfrak{j}) \end{aligned}$$

We denote $\mathcal{U}^n$, equipped with the Kolchin topology by the symbol $\mathbb{A}^n$. If V is a Kolchin closed subset of $\mathcal{U}^n$, V is a topological space in the induced topology.

Every Zariski closed set is Kolchin closed, but, not conversely. The Kolchin topology is a much larger topology than the Zariski topology. For example, every Zariski closed subset of $\mathbb{A}^1$ is finite, whereas non–finite Kolchin closed subsets abound. Indeed, there are strictly increasing chains of Kolchin closed subsets: If $\mathcal{U}$ is ordinary, with derivation operator, ∂, and, $x \in \mathcal{U}$ has derivative 1, we have the chain $\{0\} = V([y]) \subset \mathcal{K} = V([\partial y]) \subset \{ax + b : a, b \in \mathcal{K}\} = V([\partial^2 y]) \subset \{ax^2 + bx + c : a, b, c \in \mathcal{K}\} = V([\partial^3 y]) \subset ....$

Exercise 1.3. *Prove that if $\mathcal{U}$ is an ordinary ∂-field, $X = \{p(x) : p(x)$ a polynomial with coefficients in $\mathcal{K}\}$ is not Kolchin closed in $\mathbb{A}^1$.*

Note: $V\left(\sqrt{\mathfrak{i}}\right) = V(\mathfrak{i})$. E.G., in $\mathcal{U}\{y\}$, $V([y^2]) = \{0\} = V\left(\sqrt{[y^2]}\right) = V([y])$.

Call two systems of differential equations equivalent if they have the same solutions. An important motivation for Ritt in his development of differential algebra was to show that given a system of algebraic differential equations with coefficients in a differential field of meromorphic functions, there is an equivalent finite system. Ritt replaced systems of algebraic differential equations with differential polynomial ideals. He discovered early on, however, the disturbing fact that not every differential polynomial ideal is finitely differentially generated.

Example 1.4. Let $\mathcal{F}$ be an ordinary differential field. The differential ideal $\left[y^{(i)}y^{(i+1)}\right]_{i=0,1,2,...}$ in $\mathcal{F}\{y\}$ does not have a finite differential ideal basis.

Fortunately, if the given system of differential equations is

$$f_i = 0, i \in I$$

then, the system defined by the radical of the differential polynomial ideal $\mathfrak{i} = [f_i]_{i\in I}$ is equivalent to the given system. Although $\sqrt{\mathfrak{i}}$ need not be finitely differentially generated, there exists a finite subset $g_1, \ldots, g_r$ of $\mathfrak{i}$ such that $\sqrt{\mathfrak{i}} = \sqrt{[g_1, \ldots, g_r]}$. Thus, the systems $f_i = 0, i \in I$ and $g_j = 0, j = 1, \ldots, r$ are equivalent.

Theorem 1.5. *Ritt-Raudenbush Basis Theorem (1932) If* $\mathfrak{i}$ *is a radical* ∂*-ideal in* $\mathcal{U}\{y_1,\ldots,y_n\}$*, then, there exists a finite family* $(f_1,\ldots,f_r)$ *of elements of* $\mathfrak{i}$ *such that* $\mathfrak{i}=\sqrt{[f_1,\ldots,f_r]}$.

Corollary 1.6. *(radical* ∂*-Noetherianity) Every strictly ascending chain of radical* ∂*-ideals of* $\mathcal{U}\{y_1,\ldots,y_n\}$ *is finite.*

Let V be a Kolchin closed subset of $\mathbb{A}^n$. $\{f\in\mathcal{U}\{y_1,\ldots,y_n\}: f(V)=0\}$ is a radical ∂-ideal called the *defining* ∂*-ideal* of V, and, denoted by $I(V)$.

We refer to $f_1,\ldots,f_r$ as the defining ∂-polynomials of $V(\mathfrak{i})$, and, call them a *basis* of the radical ∂-ideal $\mathfrak{i}$. V is a ∂-$\mathcal{F}$-variety iff its defining differential ideal has a basis with coefficients in $\mathcal{F}$.

Theorem 1.7. *(Ritt Nullstellensatz) There is an inclusion reversing bijective correspondence between the set of Kolchin closed subsets of* $\mathbb{A}^n$ *and the set of radical* ∂*-ideals of* $\mathcal{U}\{y_1,\ldots,y_n\}$*, given by* $V\longmapsto I(V)$*,* $\mathfrak{i}\longmapsto V(\mathfrak{i})$.

$$\begin{aligned} V(I(V)) &= V \\ I(V(\mathfrak{i})) &= \sqrt{\mathfrak{i}} \\ V\subset W &\Longrightarrow I(V)\supset I(W) \\ I(V\cup W) &= I(V)\cap I(W) \\ I(V\cap W) &= \sqrt{I(V)+I(W)} \end{aligned}$$

Definition 1.8. A topological space is *Noetherian* if every strictly descending sequence of closed sets is finite.

Corollary 1.9. $\mathbb{A}^n$*, equipped with the Kolchin topology is a Noetherian topological space.*

Definition 1.10. A topological space is *reducible* if it is the union of two proper closed subsets.

Remark 1.11. A topological space X is *connected* if $\emptyset$ and X are the only subsets that are both open and closed. If X is irreducible, then, X is connected, but, not conversely. In the usual topology on the real plane, the union of the x- and, y- axes is connected, but, is not irreducible.

Which Kolchin closed subsets are irreducible?

Lemma 1.12. *A Kolchin closed subset $V \subset \mathbb{A}^n$ is irreducible if and only if $I(V)$ is prime.*

Proposition 1.13. *(Kovacic) Let $\mathcal{R}$ be a ∂-ring containing $\mathbb{Z}$, and, let $\mathfrak{i}$ be a proper ∂-ideal of $\mathcal{R}$. The following are equivalent:*

1. *Let $\Sigma \subset \mathcal{R}$ be a multiplicative set such that $\mathfrak{i} \cap \Sigma = \emptyset$. Then, a ∂-ideal of $\mathcal{R}$ maximal among all ∂-ideals of $\mathcal{R}$ containing $\mathfrak{i}$, and excluding Σ, is prime.*
2. *$\sqrt{\mathfrak{i}}$ is a ∂-ideal.*
3. *Every minimal prime ideal of $\mathcal{R}$ containing $\mathfrak{i}$ is a ∂-ideal.*

Corollary 1.14. *Let $\mathcal{R}$ be a ∂-ring containing the field $\mathbb{Q}$ of rational numbers, and, let $\mathfrak{i}$ be a proper ∂- ideal of $\mathcal{R}$. An element f of $\mathcal{R}$ is in $\sqrt{\mathfrak{i}}$ if and only if it is in every prime ∂-ideal containing $\mathfrak{i}$.*

Theorem 1.15. *Let $\mathfrak{i}$ be a proper radical ∂-ideal in $\mathcal{U}\{y_1, \ldots, y_n\}$. Then, $\mathfrak{i}$ can be written uniquely up to order as a finite intersection of minimal prime ideals. Each minimal prime ideal containing $\mathfrak{i}$ is a ∂-ideal.*

Corollary 1.16. *Let V be a Kolchin closed subset of $\mathbb{A}^n$. Then, V can be written uniquely (up to order) as a union of distinct maximal irreducible Kolchin closed subsets, called the irreducible components of V. The irreducible components of V are the zero sets of the minimal prime components of $I(V)$. If W is an irreducible subset of V, then, W is contained in a component of V.*

Exercise 1.17.

1. *Prove that if V is an irreducible Kolchin closed subset of $\mathbb{A}^n$, then, every non-empty Kolchin open subset of V is Kolchin dense in V.*
2. *The Kolchin topology is not Hausdorff: Prove that if V is an irreducible Kolchin closed subset of $\mathbb{A}^n$, and, U_1, U_2 are non-empty Kolchin open subsets of V, then, $U_1 \cap U_2 \neq \emptyset$.*

Direct products of ∂-varieties are essential to the definition of groups in the category. First, we identify the product $\mathbb{A}^r \times \mathbb{A}^s$ of the sets with $\mathbb{A}^{r+s}$: $(a, b) = (a_1, \ldots, a_r, b_1, \ldots, b_s)$. Then, we place on $\mathbb{A}^{r+s}$ the Kolchin topology.

If V is a Kolchin closed subset of $\mathbb{A}^r$, and, W is a Kolchin closed subset of $\mathbb{A}^s$, then, we place on $V \times W$ the induced Kolchin topology.

If $V = V(\mathfrak{i})$, $\mathfrak{i}$ a ∂-ideal in $\mathcal{U}\{y\}$, and, $W = V(\mathfrak{j})$, $\mathfrak{j}$ a ∂-ideal in $\mathcal{U}\{z\}$, then, $V_1 \times V_2 = V(\mathfrak{i}\mathcal{U}\{y,z\} + \mathfrak{j}\mathcal{U}\{y,z\})$. In particular, if V and, W are ∂-$\mathcal{F}$-varieties, so is their product. The differential ideal $\mathfrak{k} = \mathfrak{i}\mathcal{U}\{y,z\} + \mathfrak{j}\mathcal{U}\{y,z\}$ is generated by ∂-polynomials of the form $f(y)$ and $g(z)$.

Example 1.18. Let $\mathcal{U}$ be an ordinary differential field, with derivation operator ∂. We define ∂-subvarieties V and W of affine 1-space $\mathbb{A}^1$ as follows:

$$V = V([y' - y])$$
$$W = V([y']) .$$

$V \times W$ is the ∂-subvariety of the affine plane $\mathbb{A}^2$defined by the differential ideal $[y' - y, z'] \subset \mathcal{U}\{y,z\}$. Note the separation of variables in this ∂-variety that is closed in the product Kolchin topology. This indicates that the Kolchin topology on the product space is not the product of the Kolchin topologies.

Note that the product of two irreducible ∂-varieties is irreducible since $\mathcal{U}$ is algebraically closed.

Exercise 1.19. *Let C be the curve in $\mathbb{A}^2$ defined by the equation $y_2 = y_1^2$. It is not product closed in the Zariski topology. Show it is not product closed in the Kolchin topology.*

1.2 The ∂-coordinate ring on a ∂-variety.

Definition 1.20. Let $\mathcal{R}$ be a ring, and, $a \in \mathcal{R}$. a is *nilpotent* if there is a positive integer n such that $a^n = 0$. $\mathcal{R}$ is *reduced* if $\mathcal{R}$ has no nonzero nilpotent elements.

Let V be a Kolchin closed subset of $\mathbb{A}^n$, and, let $\mathfrak{i}$ be its defining differential ideal in $\mathcal{U}\{y\}$. Then, $\mathfrak{i}$ is a radical ∂-ideal. Therefore, the residue class ring $\mathcal{R} = \mathcal{U}\{\overline{y}\}$, where $\overline{y}$ is the n-tuple of residue classes mod $\mathfrak{i}$, is a reduced ∂-ring. Of course, we define $\partial\overline{y_i} = \overline{\partial y_i}$. If $f(\overline{y}) \in \mathcal{R}$ and, $a \in V$, we define $f(a) = F(a)$, where $F \bmod \mathfrak{i} = f$. $\mathcal{R}$ is called the ∂-*coordinate ring* of V. We often denote it by $\mathcal{U}\{V\}$. $\mathcal{R}$ is an integral domain iff $\mathfrak{i}$ is prime. (We will feel free to unbar the residue classes, and write simply y_i.)

There is a bijective correspondence between the ∂-ideals of $\mathcal{R}$ and the ∂-ideals of $\mathcal{U}\{y\}$ containing $\mathfrak{i}$, given by $\mathfrak{j} \longmapsto \mathfrak{j} \bmod \mathfrak{i}$. So, there is a bijective correspondence between the radical ∂-ideals of $\mathcal{R}$ and the Kolchin closed subsets of V, given by $\mathfrak{j} \longmapsto V(\mathfrak{j})$. We call $\mathfrak{j} \bmod \mathfrak{i}$ the defining ∂-ideal of $V(\mathfrak{j})$ in $\mathcal{R}$. $\mathfrak{j}$ is (radical) prime iff $\mathfrak{j} \bmod \mathfrak{i}$ is (radical) prime.$\mathfrak{j}$ is a minimal prime containing $\mathfrak{i}$ iff $\mathfrak{j} \bmod \mathfrak{i}$ is a minimal prime of $\mathcal{R}$. The minimal primes of $\mathcal{R}$ are the defining ∂-ideals in $\mathcal{R}$ of the irreducible components of $\mathcal{R}$.

Suppose V is irreducible. The elements of the quotient field of $\mathcal{R} = \mathcal{U}\{V\}$ are called *∂-rational functions* on V. It is denoted by $\mathcal{U}\langle V\rangle$. f is a ∂-rational function on V iff there exists $p, q \in \mathcal{R}$ such that $f = \frac{p}{q}$. If $g = \frac{r}{s} \in \mathcal{U}\langle V\rangle$ then, $f = g$ iff $PS - QR \in \mathfrak{i}$, where $P \bmod \mathfrak{i} = p$, $Q \bmod \mathfrak{i} = q$, $R \bmod \mathfrak{i} = r$, $S \bmod \mathfrak{i} = s$, $T \bmod \mathfrak{i} = t$.

Let $a \in V$, and, $f \in \mathcal{U}\langle V\rangle$. Then, f is *defined at a* iff there exist $p, q \in \mathcal{R}$ such that $f = \frac{p}{q}$, and, $q(a) \neq 0$. The domain $D(f) = \{a \in V : f \text{ is defined at } a\}$ is a non-empty (hence Kolchin dense) subset of V. f is *everywhere defined* if $D(f) = V$. The set of *everywhere defined* ∂-rational functions on V is a ∂-subring of $\mathcal{U}\langle V\rangle$ containing $\mathcal{U}\{V\}$, denoted by $\widehat{\mathcal{R}}$. So, $\mathcal{R} \subset \widehat{\mathcal{R}} \subset \mathcal{U}\langle V\rangle$. In a scheme-theoretic approach to differential algebraic geometry, $\widehat{\mathcal{R}}$ is the ring of global sections of the structure sheaf on V.

Remark 1.21.

1. In contrast to affine algebraic geometry, $\mathcal{R} \neq \widehat{\mathcal{R}}$. For example, set $V = V([\partial y - y])$, where ∂ is the derivation operator on the ordinary differential field $\mathcal{U}$. Let $x \in \mathcal{U}$ have derivative 1. Then, $V = \{ce^x : c \in \mathcal{K}\}$. $\mathcal{U}\{V\} = \mathcal{U}[y]$. The differential rational functions $\frac{1}{y-c}$, $c \in \mathcal{K}$, $c \neq 0$, are everywhere defined on V, and, are not in $\mathcal{U}\{V\} = \mathcal{U}[\mathcal{V}]$. Note that this shows that $\widehat{\mathcal{R}}$ is not finitely ∂-generated over $\mathcal{U}$.

2. If $f \in \widehat{\mathcal{R}}$, the Noetherianity of the Kolchin topology implies that there exist a finite number of denominators q such that $f = \frac{p}{q}$, and, $\forall a \in V$, there exists q in this finite set such that $q(a) \neq 0$. However, in contrast to affine algebraic geometry, where we may always take $q = 1$, we may need more than one denominator.

On p. 137 of his book *Differential algebraic groups,* (1985) Kolchin gives the following example: Let $\mathcal{U}$ be an ordinary differential field with derivation operator ∂. Write $\partial y = y', \partial^2 y = y''$, etc. Let V be the ∂-subvariety of $\mathbb{A}^2$, defined by the equation $(y_1'+1)y_2'-y_1y_2''$. Then, V is irreducible. Set $f=\frac{y_2'}{y_2}$. Then, $f=\frac{y_2''}{y_1+1}$. f is everywhere defined, but, we need both denominators to define it everywhere. A proof was given by Kovacic (2002).

Remark 1.22. If V is reducible, with irreducible components $V_1,\ldots,V_k$, a k-tuple $f=(f_1,\ldots,f_k)$, where f_i is a differential rational function on V_i, is called a *∂-rational function* on V. $D(f)$ is a Kolchin open dense subset of V. So, the ring of differential rational functions on V is the direct product of the differential fields of differential rational functions on its irreducible components. It is also the complete ring of fractions of its coordinate ring.

1.3 Differential rational maps and morphisms.

Let V be a ∂-subvariety of $\mathbb{A}^r$ and let W be a ∂-subvariety of $\mathbb{A}^s$. A *differential rational map* $f:V-->W$ is an s-tuple $f=(f_1,\ldots,f_s)$ of differential rational functions on V such that f maps its domain $D(f)$ into W. If the coordinates of f are in $\widehat{\mathfrak{R}}$, then, we call f a *morphism*. Differential rational maps are our interpretation of the Bäcklund transformations of physics (variational formalism).

The next theorem, proved in algebraic geometry by Claude Chevalley, and, later, in differential algebraic geometry by Abraham Seidenberg, is important for geometry. André Weil refers to it as a "device...that finally eliminates from algebraic geometry the last traces of elimination theory." It is a key theorem in contemporary logic (model theory).

Theorem 1.23. *Chevalley-Seidenberg Let V and W be ∂-varieties, and, let $f:V-->W$ be a differential rational map. The image of $D(f)$ contains a set that is Kolchin open and dense in the closure of the image of $D(f)$. If V and W are ∂-$\mathfrak{F}$-varieties, so is the closure of $D(f)$.*

Definition 1.24. An irreducible ∂-variety V whose field of differential rational functions has finite transcendence degree over $\mathfrak{U}$ is called *finite-dimensional.* $\dim V = tr\deg_{\mathfrak{U}}\mathfrak{U}\langle V\rangle$.

The ∂-varieties of the following example satisfy partial differential equations, and, are not finite-dimensional. However, the fibers of the differential rational map are finite-dimensional. Following Igonin (2005), we might call the domain variety a covering variety of the image variety.

I would like to close the sections on differential rational functions and maps with an example: the transformation of Burgers'equation into the Heat Equation. Modern interest in integrable systems connected with non-linear differential equations was sparked by the discovery of solitons by Kruskal and Zabusky (1978), and, the subsequent close study of the barely non-linear KdV equation $\partial_t y = \partial_x^3 y + 6y\partial_x y$. The awareness of solitons dates back, however, to a horseback ride in 1834.

The Scottish nautical engineer, John Scott Russell writes:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddensly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on ...preserving its original figure some thirty feet long and a foot to a foot and a half in height.

Our example is less exotic. It illustrates the use of Rosenfeld coherence in proving surjectivity of differential rational morphisms, and, along the way, connects, at least in this particular example, Rosenfeld coherence with integrability conditions.

Example 1.25. Kaup (1980) This example of a Bäcklund transformation of the heat equation into Burgers' equation illustrates the the Rosenfeld coherence property, and, its connection with the integrability conditions mentioned by Sally Morrison. Let $\partial = \{\partial_x, \partial_t\}$. Let V be the solution set in $\mathbb{A}^1$ of the heat equation

$$\partial_t y + \partial_x^2 y = 0,$$

and, let W be the solution set of Burgers' equation (Johannes Martinus Burgers, 1895-1981), from fluid dynamics:

$$\partial_t y + \partial_x^2 y + 2y\partial_x y = 0.$$

The *Cole-Hopf transformation* $\ell\partial_x = \frac{\partial_x y}{y}$ maps its domain $V\backslash\{0\}$ into W (Exercise).

Using Rosenfeld coherence, we will show that the image of the *domain* of the Cole-Hopf transformation equals W, *i.e.*, the transformation is *surjective on its domain.* This transformation of Burgers' equation into the heat equation helps physicists find exact solutions of Burgers' equation by finding exact solutions of the heat equation.

Let z satisfy Burgers' equation. We want to show that the following system of equations and inequations has a solution. Set $\mathfrak{l} = [g, h]$, where

$$\begin{aligned} h &= \partial_x^2 y + \partial_t y \\ g &= \partial_x y - zy \\ \partial_t z + \partial_x^2 z + 2z\partial_x z &= 0 \\ y &\neq 0 \end{aligned}$$

Then, $\mathfrak{l}$ is linear, hence is prime. To solve our problem, we will find a good characteristic set for $\mathfrak{l}$ with respect to some ranking. As Sally Morrison and William Sit pointed out, a characteristic set of a differential polynomial ideal reduces the ideal to 0. Therefore, y is not in the ideal. We choose an orderly ranking with $\partial_t < \partial_x$. The leader of h is $\partial_x^2 y$. The leader of g is $\partial_x y$.

Reduce h with respect to g (Exercise). Get: $h = \partial_x g + zg + r$, where $r = \partial_t y + (z^2 + \partial_x z)\, y$.

$$\begin{aligned} r &= \partial_t y + \left(z^2 + \partial_x z\right) y \\ g &= \partial_x y - zy, \\ \partial_t z + \partial_x^2 z + 2y\partial_x z &= 0 \end{aligned}$$

Now, $\mathfrak{l} = [g, h] = [g, r]$. The set $A : g, r$ is autoreduced, and, the lowest common derivative of the leaders is $\partial_x \partial_t y$. We now compute the Rosenfeld s-polynomial $s = \partial_x r - \partial_t g$.

$$s = \left(z^2 + \partial_x z\right) \partial_x y + \left(\partial_x^2 z + 2z\partial_x z\right) y + z\partial_t y + (\partial_t z)y.$$

Now, A is coherent iff s is in the ideal (g, r). Find Burgers!

$$\begin{aligned} s - \left(z^2 + \partial_x z\right) g &= zr + b(z)y \\ s &= b(z)y + zr + (z^2 + \partial_x z)g \end{aligned}$$

So, A is coherent iff $b\,(z) = 0$. Burgers' equation is a coherence condition on the autoreduced set A. It follows, since $\mathfrak{l}$ is prime, that A is a characteristic set for $\mathfrak{l}$. In particular, if a differential polynomial is reduced with respect to A, it cannot be in $\mathfrak{l}$.

So, $\ell\partial_x$ is a surjective map from its domain $V\setminus\{0\}$, the set of nonzero solutions of the heat equation onto W, the solution set of Burgers' equation. The fiber $\ell\partial^{-1}(z), z \in W$, is the set of solutions of the pair of differential equations

$$\frac{\partial_x y}{y} = z$$
$$\frac{\partial_t y}{y} = -(z^2 + \partial_x z)$$

This pair of differential equations has a solution $\Longleftrightarrow b(z) = 0$. So, Burgers'equation is the integrability condition in the classical sense. The proof in differential algebraic geometry uses Rosenfeld coherence. Note that the fiber of $\ell\partial_x$ is a ∂-variety that is finite-dimensional, of dimension 1. Each fiber gives us a finite-dimensional ∂-subvariety of the infinite-dimensional ∂-variety $V(h)$.

2 Affine differential algebraic groups.

Definition 2.1.

1. Let G be a group. G is an *affine ∂-group* if, for some n, G is a Kolchin closed subset of $\mathbb{A}^n$, and, the group multiplication $G\times G \to G$ and inversion $G \to G$ are morphisms of ∂-varieties. If multiplication and inversion are defined over $\mathcal{F}$, and, the identity element 1 has coordinates in $\mathcal{F}$, we call G a ∂-$\mathcal{F}$-group.

2. Let G be a ∂-group. There is a unique component G^0 containing 1, which is a normal ∂-subgroup whose cosets are the irreducible components of G. In particular, the components are mutually disjoint, and, of course, equal the connected components of G.

3. If G and G' are affine ∂-groups, a homomorphism $f : G \to G'$ of ∂-groups is a group homomorphism, and, a ∂-variety morphism.

Remark 2.2. Every affine algebraic group is an affine ∂-group.

1. Let G be a ∂-group, and, let $a \in G$. An important homomorphism of ∂-groups is the *inner* automorphism $x \longmapsto axa^{-1}$.

2. The additive group $\mathbb{G}_a^n$ of $\mathcal{U}^n$ is a ∂-group. Let $L_i, i = 1, \ldots, n \in \mathcal{U}[\partial]$, the non-commutative ring of linear differential operators (non-commuting polynomials) in $\partial_1, \ldots, \partial_m$. Let l be the surjective ∂-endomorphism of $\mathbb{G}_a^n$ with coordinate functions $L_1, \ldots, L_n$. Every ∂-endomorphism of $\mathbb{G}_a^n$ is in this form.

Let $f : G \to G'$ be a ∂-group homomorphism. It is easy to see that $\ker f$ is a normal ∂-subgroup of G. The fact that $\mathrm{im} f$ is a subgroup of G' follows from the Chevalley-Seidenberg Theorem. If G and G' are ∂-$\mathcal{F}$-groups, and, f is defined over $\mathcal{F}$, then, $\ker f$, and, $\mathrm{im}\ f$ are ∂-$\mathcal{F}$-groups.

If $f : G \to G'$ is a homomorphism, $f(G^0)$ is the identity component of $\mathrm{im}\ f$. So, the image of a connected ∂-group is connected.

Notation 2.3. If G is an affine ∂-$\mathcal{F}$-group, the subgroup of points in G with coordinates in a ∂-extension field $\mathcal{G}$ of $\mathcal{F}$ in $\mathcal{U}$ is denoted by $G(\mathcal{G})$.

As we saw in Jerry Kovacic's talks, $GL_n(\mathcal{U})$ is a Zariski open set in $\mathbb{A}^{n^2} = M_n(\mathcal{U})$. We close it up by identifying it with the ∂-subgroup of $\mathbb{A}^{n^2+1}$ defined by the equation $z \det y = 1$, where $y = (y_{ij})_{i,j=1,\ldots,n}$.is a matrix of differential indeterminates. The coordinate ring of $GL_n(\mathcal{U})$ is $\mathcal{U}\left\{(y_{ij}), \frac{1}{\det}\right\}$. When we give the defining equations of a ∂-subgroup of $GL_n(\mathcal{U})$, we will omit the equation $z \det y = 1$, since it is universally satisfied. We denote $GL_1(\mathcal{U})$ by $\mathbb{G}_m$.

Remark 2.4. Let $\mathbb{R}$ be the field of real numbers, and, $\mathcal{U}$ a differential closure of the quotient field $\mathcal{F}$ of the ring of real analytic functions on the unit circle S^1. $GL_n(\mathcal{F}) = GL_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{F}$ can be conceptualized as the loop group $Map(S^1, GL_n(\mathbb{R}))$.

An affine ∂-group G is *linear* if there is an isomorphism from G into some $GL_n(\mathcal{U})$. *Every affine algebraic group is linear.*

Example 2.5. Let $\mathcal{U}$ be an ordinary differential field with derivation operator ∂.

1. The only algebraic group structure on the affine plane $\mathbb{A}^2$ (over any algebraically closed field of characteristic 0) is the additive group $\mathbb{G}_a \times \mathbb{G}_a$. The following are the ∂-group structures on $\mathbb{A}^2$ (up to isomorphism):

$$(u_1, u_2)(v_1, v_2) = \left(u_1 + v_1, u_2 + v_2 + \sum_{i<j} u_1^{(i)} v_1^{(j)}\right)$$

where $u^{(i)} = \partial^i u$. (The sum $\sum\limits_{i<j} u_1^{(i)} v_1^{(j)}$ is a differential polynomial 2-cocycle from $\mathbb{G}_a$ into $\mathbb{G}_a$.) These groups are *unipotent* linear ∂-groups. The group is "unipotent" since it can be embedded in a group of upper triangular matrices with $1'$s on the diagonal.

For example, the group $(u_1, u_2)(v_1, v_2) = \left(u_1 + v_1, u_2 + v_2 + u_1 v_1' + u_1' v_1^{(2)}\right)$ is isomorphic to a ∂-group of 4×4 unipotent matrices:

$$\begin{pmatrix} 1 & u_1 & u_1' & u_2 \\ 0 & 1 & 0 & u_1' \\ 0 & 0 & 1 & u_1^{(2)} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If $\mathcal{U}$ is an ordinary differential field, the logicians Pillay and Kowalski proved that every ∂-group structure with underlying variety $\mathbb{A}^n$ is a unipotent linear group.

2. (Cassidy-Kovacic) Let $t \in \mathcal{U}$, transcendental over $\mathbb{Q}$, with $\partial t = 1$. Set G equal to the Kolchin closed subset of $\mathbb{A}^2$, defined by the differential equations:

$$\begin{aligned} y_2 (y_1 - 1)^2 &= t^2 \left(y_1^3 + a y_1 y_2^2 + b y_2^3\right) \\ \partial y_1 &= \frac{1}{t}\left(y_1^2 - y_1\right) \\ y_1 \partial y_2 - y_2 \partial y_1 &= 0. \end{aligned}$$

where $a, b \in \mathbb{Q}$, $b \neq 0, 4a^3 + 27b^2 \neq 0$. G is a connected commutative ∂-$\mathbb{Q}(t)$-group. $0 = (0,0)$. Note that $y_2 = 0$ implies $y_1 = 0$, and, that $2y_1 - 1$ never vanishes on G (Exercise). $-(y_1, y_2) = \left(\frac{y_1}{2y_1 - 1}, \frac{y_2}{2y_1 - 1}\right)$.

Coordinatize $G \times G$ by $(y_1, y_2), (z_1, z_2)$. On the open set $y_1 y_2 (y_1 z_2 - y_2 z_1) \neq 0$, the addition law is given by (v_1, v_2), where

$$\begin{aligned} v_1 &= \frac{1}{t^2}\left(\frac{y_1 z_2 - z_1 y_2 + y_2 - z_2}{y_1 z_2 - y_2 z_1}\right)^2 - \frac{y_1 z_2 + y_2 z_1}{y_2 z_2}. \\ v_2 &= -\frac{1}{t^3}\left(\frac{y_1 z_2 - y_2 z_1 + y_2 - z_2}{y_1 z_2 - y_2 z_1}\right)^3 + \\ &\quad + \frac{1}{t}\cdot\left(\left(\frac{y_1 z_2 - y_2 z_1 + y_2 - z_2}{y_1 z_2 - y_2 z_1}\right)\left(\frac{y_1 z_2 + y_2 z_1}{y_2 z_2}\right) - \left(\frac{z_1 - y_1}{y_1 z_2 - y_2 z_1}\right)\right) \end{aligned}$$

Perhaps you realized that the affine ∂-$\mathbb{Q}(t)$-group G is an embedding in the affine $\mathcal{U}$-plane of the ∂-subgroup $E(\mathcal{K})$ consisting of the constant points of the elliptic curve in the projective $\mathcal{U}$-plane with affine equation $y_2^2 = y_1^3 + ay_1 + b$ The embedding is a rational isomorphism defined over $\mathbb{Q}(t)$. G is not linear. G has no non-trivial linear representations. The proof mainly consists in counting the number of points of order dividing n.

The group of example 1 is not finite-dimensional if the transcendence degree of $\mathcal{U}$ over $\mathbb{Q}$ is infinite. The group of example 2 is finite-dimensional.

Problem 2.6. *1. Characterize all affine non-linear differential algebraic groups*

2. When $\mathcal{U}$ is an ordinary differentially closed differential field, the logicians Hrushovski, Sokolovich, and Pong, have shown that all finite-dimensional ∂-groups can be embedded in the affine $\mathcal{U}$-line. Characterize all affine differential algebraic groups with no non-trivial linear representations. Are they all commutative? Must they be finite-dimensional? Are they all obtained by affine ∂-embeddings of abelian varieties?

3 Linear differential algebraic groups.

For ease of exposition, we reluctantly assume that $\mathcal{U}$ is an ordinary differential field with derivation operator ∂, and field $\mathcal{K}$ of constants. As usual, we often denote ∂y by $y', \partial^2 y$ by y'',....Everything in the following discussion has a parallel in the partial case.

Let G be an affine ∂-group, with coordinate ring $\mathcal{U}\{G\} = \mathcal{R}$, and, ring of everywhere defined differential rational functions $\widehat{\mathcal{R}}$. Let $a \in G$. We define a ∂-automorphism over $\mathcal{U}$ of $\widehat{\mathcal{R}}$ by

$$(fa)(b) = f(ba),\ f \in \widehat{\mathcal{R}}, b \in G.$$

If $G = GL_n(\mathcal{U})$, then, $(y_{ij})a = \sum_{k=1}^{n} y_{ik}a_{kj}$. The map $\rho : G \to Aut_\partial\widehat{\mathcal{R}}$ is an injective homomorphism of groups, called the *regular representation* of G. If G is a subgroup of $GL_n(\mathcal{U})$, then, for all $a \in G$, for all $f \in \mathcal{R}$, fa is in $\mathcal{R}$. In this case, we refer to the restriction of the regular representation to $\mathcal{R}$ as the regular representation of G.

Theorem 3.1. *Let G be a linear ∂-group, and, let N be a normal ∂-subgroup. There exists a linear ∂-group G', and, a surjective homomorphism $q : G \rightarrow G'$, with kernel N such that if H is a linear ∂-group, and, $f : G \rightarrow H$ is a homomorphism whose kernel contains N, there is a homomorphism $g : G' \rightarrow H$ such that $g \circ q = f$.*

G' is called the *quotient group* of G by N, and, is denoted by G/N.

Let G be a ∂-subgroup of $GL_n(\mathcal{U})$, and, let H be a ∂-subgroup of G. $\mathcal{S} = \{f \in \mathcal{R} : fa = f \ \forall h \in H\}$ is a ∂-subalgebra over $\mathcal{U}$ of $\mathcal{R}$ called the *ring of invariant differential polynomal functions* of H.

Recall: Not every differential polynomial ideal is finitely differentially generated (J. F. Ritt). Boris Weisfeiler ingeniously adapted Ritt's example to show that, in contrast to algebraic group theory, the ring of invariants in $\mathcal{U}\{y\}$ (under the regular representation of $\mathbb{G}_m$) of the *finite* subgroup $\{\pm 1\}$ is not finitely ∂-generated. If we replace $\mathcal{U}\{y\}$ by the coordinate ring $\mathcal{U}\left\{y, \frac{1}{y}\right\}$, which is a Hopf algebra, this cannot happen..

Theorem 3.2. *If G is a ∂-subgroup of $GL_n(\mathcal{U})$, and, N is a normal ∂-subgroup of G, the subring $\mathcal{S} \subset \mathcal{R}$ of invariant differential polynomial functions of N is finitely ∂-generated.*

Corollary 3.3. *Let $G' = G/N$. Then, the coordinate ring $\mathcal{U}\{G'\}$ is isomorphic over $\mathcal{U}$ to the ∂-algebra over $\mathcal{U}$ of invariants of N in $\mathcal{U}\{G\}$.*

Problem 3.4. *Prove that quotients of affine ∂-groups exist. Are they also affine?*

3.1 The differential algebraic subgroups of $\mathbb{G}_m$.

The only proper algebraic subgroup of $\mathbb{G}_a$ is $\{0\}$. If G is a ∂-subgroup, there exist linear differential operators $L_1, \ldots, L_r \in \mathcal{U}[\partial]$ such that $G = \bigcap_{i=1}^{r} \ker L_i$.

The only proper algebraic subgroups of $\mathbb{G}_m$ are the groups G of n^{th} roots of unity. The defining equation of G is $y^n - 1 = 0$. $f(y) = y^n - 1$ is also the defining differential polynomial invariant of G. The next theorem tells us that all the ∂-subgroups of $\mathbb{G}_m$ are defined by a single differential polynomial invariant in $\mathcal{U}\{\mathbb{G}_m(\mathcal{U})\} = \mathcal{U}\left\{y, \frac{1}{y}\right\} = \mathcal{R}$.

Theorem 3.5. *Let G be a ∂-subgroup of $\mathbb{G}_m$. Then, there is a differential polynomial $f \in \mathcal{R}$ such that $G = \{a \in \mathbb{G}_m : fa = f\}$. Moreover, the defining ∂-ideal of G in $\mathcal{R}$ is equal to $[f]$.*

What do the defining differential polynomial invariants of infinite ∂-subgroups of $\mathbb{G}_m$ look like? We met the homomorphism that defines all the invariants of positive order in Jerry's talks on Picard-Vessiot theory.

We define a surjective homomorphism $\ell\partial : \mathbb{G}_m \to \mathbb{G}_a$ by the formula $\ell\partial a = \frac{\partial a}{a}$. $\ker \ell\partial = \mathbb{G}_m(\mathcal{K})$. So, we have a short exact sequence

$$1 \to \mathbb{G}_m(\mathcal{K}) \to \mathbb{G}_m \stackrel{\ell\partial}{\to} \mathbb{G}_a \to 0$$

Note that $\mathbb{G}_m(\mathcal{K})$ is the Kolchin closure of the torsion group of $\mathbb{G}_m$.

Theorem 3.6. *Let f be the defining differential polynomial invariant of a proper infinite ∂-subgroup G of $\mathbb{G}_m$. Then, there is a linear homogeneous differential operator $L = \partial^n + a_1\partial^{n-1} + \ldots + a_n$ in $\mathcal{U}[\partial]$ such that $f = L(\ell\partial y)$.*

Example 3.7. $f(y) := \partial(\ell\partial y) = \partial\left(\frac{\partial y}{y}\right)$. f is a surjective ∂-homomorphism from $\mathbb{G}_m$onto $\mathbb{G}_a$, with kernel G. We have a short exact sequence

$$1 \to G \to \mathbb{G}_m \stackrel{\partial\circ\ell\partial}{\to} \mathbb{G}_a \to 0$$

The group G leaving f invariant is the group $c_0 e^{c_1 t}$, where $\partial t = 1$, and, $\partial(e^t) = e^t$. G can also be defined as the set of all solutions in $\mathbb{G}_m$ of the second order differential equations $y\partial^2 y - (\partial y)^2 = 0$.

Corollary 3.8. *Every infinite proper ∂-subgroup G of $\mathbb{G}_m$ contains $\mathbb{G}_m(\mathcal{K})$, and, is connected.*

Proof. That $G \supset \mathbb{G}_m(\mathcal{K})$ is clear. That G is connected follows from the fact that $[L(\ell\partial y)]$ is prime. □

Remark 3.9. $\ell\partial$ defines a 1-1 correspondence between the infinite ∂-subgroups of $\mathbb{G}_m$ and the ∂-subgroups of $\mathbb{G}_a$. There is a parallel of this for the ∂-subgroup lattice of an elliptic curve $E(\mathcal{U})$ viewed as a ∂-group. If $E(\mathcal{U})$ does not descend to constants, $\ell\partial$ is replaced by the Manin homomorphism, whose kernel is finite-dimensional, of dimension $\dot{2}$.

3.2 Simple differential algebraic groups.

Definition 3.10. 1. Let G be an affine ∂-group, and V a ∂-variety. An *action* of G on V is a triple (G, V, f), where f is a morphism $G \times V \to V$, sending (z, x) to $zx = f(z, x)$, such that

$$1x = x$$
$$z_1(z_2 x) = (z_1 z_2)x$$

$z_1, z_2 \in G, x \in V$.

2. V is a *homogeneous space* for G if the action is transitive,*i.e.*, for every$x$, $y \in V$ there is $z \in G$ such that $zx = y$. A homogeneous space V is a *torsor under* G if there is a unique such z.

3. Let $x \in V$. The set of all $z \in G$ such that $zx = x$ is a ∂-subgroup of G called the *isotropy* group of x.

Definition 3.11. An infinite algebraic group is *simple if it is not commutative, and, every proper normal algebraic subgroup is finite. Similarly, a* ∂-group G is *simple* if it is not commutative, and, every proper normal ∂-subgroup is finite. In particular, G is connected.

Theorem 3.12. *(Pillay) Every simple ∂-group is linear.*

Definition 3.13. A *Chevalley group* is a simple algebraic group that is defined over $\mathbb{Q}$.

Theorem 3.14. *(The classification of the simple ∂-groups) Let G be a simple ∂-group. There exists a simple Chevalley group H such that G is isomorphic to $H(\mathcal{U})$ or to $H(\mathcal{K})$, the group of matrices in G with entries in $\mathcal{K}$.*

Corollary 3.15. *Every simple ∂-group G can be realized as a simple Chevalley group $H(\mathcal{U})$ or as a simple Chevalley group $H(\mathcal{K})$.*

For an arithmetic analogue that replaces the derivation ∂ with a nonlinear operator on a p-adic ring, see Buium (1998).

3.2.1 The Zariski dense ∂-subgroups of $SL_n(\mathcal{U})$.

The first step in the proof of the classification theorem of the simple ∂-groups G entails embedding G in a simple Chevalley group G as a Zariski dense ∂-subgroup. Conversely, every Zariski dense ∂-subgroup of a simple Chevalley group is simple. To illustrate the second step, which describes the proper Zariski dense ∂-subgroups of a simple Chevalley group H, we take $H = SL_n(\mathcal{U})$.

Definition 3.16. Let k be a field. A vector space $\mathfrak{g}$ over k is a *Lie algebra* if there is a bilinear map $m : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, sending (a, b) to $[a, b]$ such that:

1. $[a, a] = 0$. antisymmetry $\Longrightarrow [b, a] = -[a, b]$
2. $[a, [b, c]] + [b, [c, a] + [c, [a, b]] = 0$ Jacobi identity

The vector space $M_n(\mathcal{U})$ is a Lie algebra over $\mathcal{U}$. If A and B are matrices, $[A, B] = AB - BA$. It is denoted by $g\ell_n(\mathcal{U})$. It is the Lie algebra of matrices of $GL_n(\mathcal{U})$. The Lie subalgebra that interests us is $s\ell_n(\mathcal{U}) = \{A \in s\ell_n(\mathcal{U}) : tr(A) = 0\}$. It is the Lie algebra of matrices of $SL_n(\mathcal{U})$.

$SL_n(\mathcal{U})$ acts on $s\ell_n(\mathcal{U})$ by the *adjoint action*: $Ad(Z) : A \longmapsto ZAZ^{-1}, Z \in SL_n(\mathcal{U})$,$A \in s\ell_n(\mathcal{U})$. The adjoint action maps $SL_n(\mathcal{U})$ into the automorphism group of the Lie algebra $s\ell_n(\mathcal{U})$. Its kernel is the center $\{\varpi \ 1_n : \omega^n = 1\}$.

In differential algebraic geometry, as well as in Picard-Vessiot theory, there is another action of $SL_n(\mathcal{U})$ on $s\ell_n(\mathcal{U})$, called the *gauge action*. It is built from the *logarithmic derivative* morphism.$\ell\partial : GL_n(\mathcal{U}) \to g\ell_n(\mathcal{U})$:

$$\ell\partial(Z) = \partial Z \ Z^{-1}.$$

If $Z = (z_{ij})$, $\partial Z = (\partial z_{ij})$. Now, $tr(\ell\partial(Z)) = \ell\partial(\det(Z))$ (Exercise). So, $\ell\partial$ maps $SL_n(\mathcal{U})$ into $s\ell_n(\mathcal{U})$. One can show that it is surjective since $\mathcal{U}$ is differentially closed.

Although $\ell\partial$ is a morphism of ∂-varieties, it is not a homomorphism of groups.

$$\ell\partial(Z_1 Z_2) = \ell\partial(Z_1) + Z_1 \ell\partial(Z_2) Z_1^{-1}$$

$\ell\partial$ is what we call a *crossed homomorphism* (1-cocycle), with respect to the adjoint action. Thus, it has a kernel, which is $SL_n(\mathcal{K})$.

$\ell\partial$ gives rise to the gauge action of the ∂-group $SL_n(\mathcal{U})$ on its Lie algebra $s\ell_n(\mathcal{U})$.

$$A \longmapsto ZAZ^{-1} + \ell\partial(Z) = \ell\partial(Z) + ZAZ^{-1}.$$

Since $\mathcal{U}$ is differentially closed, $s\ell_n(\mathcal{U})$ is a homogeneous space for the special linear group under this action. If we restrict the actors and actees to an arbitrary base field $\mathcal{F}$, the action may no longer be transitive. The orbits then play an important role in Picard-Vessiot theory.

Remark 3.17. Let V be a finite-dimensional vector space. The *affine group* of V is the group of all transformations $v \longmapsto T(v)+w$, where T is linear and $w \in V$. The gauge action maps $SL_n(\mathcal{U})$ into the affine group of the *vector space* $s\ell_n(\mathcal{U})$ over $\mathcal{U}$. The affine group is a linear group, and, so, as in the case of the adjoint action, the gauge action gives us a linear representation of $SL_n(\mathcal{U})$– this time as a differential algebraic group, rather than as an algebraic group. What is the kernel of this representation? Setting $A = 0$, we see that $Z \in SL_n(\mathcal{K})$. But, then, Z is in ker *Ad*, which is the center of $SL_n(\mathcal{U})$. So, the adjoint action and the gauge action have the same kernel.

Let $A \in s\ell_n(\mathcal{U})$. The isotropy group G of A in $SL_n(\mathcal{U})$ under the gauge action is defined by the differential equations

$$\det Y = 1$$
$$A = YAY^{-1} + \partial Y\, Y^{-1}$$

The second equation is a matrix equation, which is usually written $\partial Y = AY - YA = [A, Y]$, and, is called a *Lax equation*. Lax equations play an important part in the isomonodromy approach to Painlevé theory. The equation gives rise to n^2 linear homogeneous differential polynomial equations. It is easy to see that the isotropy group of A is a proper Zariski dense ∂-subgroup of $SL_n(\mathcal{U})$.

The main step in the classification theorem is the following theorem. Recall that a ∂-$\mathcal{F}$-subgroup of $GL_n(\mathcal{U})$ is a ∂-subgroup that is defined over $\mathcal{F}$.

Theorem 3.18. *Let G be a proper Zariski dense ∂-$\mathcal{F}$-subgroup of $SL_n(\mathcal{U})$ Then, there is a matrix $A \in s\ell_n(\mathcal{F})$ such that G is the isotropy group of A under the gauge action.*

Corollary 3.19. *Let G be a proper Zariski dense ∂-$\mathcal{F}$-subgroup of $SL_n(\mathcal{U})$. Then, there is a Picard-Vessiot extension $\mathcal{G}$ of $\mathcal{F}$ and a matrix $T \in s\ell_n(\mathcal{G})$ such that $T^{-1}GT = SL_n(\mathcal{K})$.*

Proof. Since the logarithmic derivative morphism is surjective, there is a matrix $T \in SL_n(\mathcal{U})$ such that $\ell\partial(T) = A$. Indeed, we can find such a matrix T generating a Picard-Vessiot extension of $\mathcal{F}$. Let $Z \in G$. We will not prove here that we can find a matrix T that is Picard-Vessiot over $\mathcal{F}$.

$$\begin{aligned}\ell\partial\left(T^{-1}ZT\right) &= \ell\partial\left(T^{-1}\right) + T^{-1}\ell\partial\left(ZT\right)T \\ &= T^{-1}(-\ell\partial\left(T\right) + \ell\partial\left(Z\right) + Z\ell\partial\left(T\right)Z^{-1})T \\ &= T^{-1}\left(-A + \ell\partial\left(Z\right) + TAT^{-1}\right) \\ &= 0.\end{aligned}$$

Therefore, $T^{-1}ZT \in SL_n(\mathcal{K})$. Similarly, one can show that $TSL_n(\mathcal{K})T^{-1}$ is the isotropy group of $A = \ell\partial(T)$ under the gauge action of $SL_n(\mathcal{U})$ on $s\ell_n(\mathcal{U})$. □

4 The Action of SL_2 on Riccati varieties.

This is the story of configurations of 4 points on the projective line. It is a plain and casual narrative, intending to show the connection between the gauge action of differential equations theory, and, projective linear transformations, and, between circles in the extended complex plane and Riccati varieties.

You may ask what 4 points on the projective line have to do with simple differential algebraic groups. Have patience, and you will find out.

For an engrossing, clearly written, modern treatment of the configuration space of 4 points on the complex projective plane, see Masaaki Yoshida, *Hypergeometric Functions, My Love*, 1997. Yoshida dedicates his book to his cats, and, his "dog, Fuku, who came to my house from nowhere to live with me."

Recall the Riccati equation from the lectures on Picard-Vessiot theory

$$R(a_0, a_1, a_2): \qquad y' = a_0 + a_1 y + a_2 y^2, (a_0, a_1, a_2) \in \mathcal{U}^3,$$

A ∂-subvariety of $\mathbb{A}^1(\mathcal{U})$ defined by a Riccati equation is called a *Riccati variety*. Note that there is a 1-1 correspondence between the set of Riccati equations and $\mathcal{U}^3$.

The differential algebraic geometry of a Riccati variety is interesting. We embed it in the projective line $\mathbb{P}^1(\mathcal{U})$.

Let k be a field. $\mathbb{P}^1(k)$ is the set of equivalence classes of pairs

$$(u_1, u_2) \in \mathbb{A}^2(k) \qquad (u_1, u_2) \neq 0,$$

under the equivalence relation

$$(u_1, u_2) \sim (\lambda u_1, \lambda u_2) \qquad \lambda \in k, \lambda \neq 0.$$

The class of (u_1, u_2) is denoted by $[u_1, u_2]$.

A polynomial $P \in k[y_1, y_2]$ is *homogeneous* if there is a positive integer d such that for all $\lambda \in k$, $P(\lambda y_1, \lambda y_2) = \lambda^d P(y_1, y_2)$. A subset of $\mathbb{P}^1(k)$ is *Zariski closed* if it is the set of zeros of a finite set of homogeneous polynomials in $k[y_1, y_2]$.

Similarly, a differential polynomial $P \in \mathcal{U}\{y_1, y_2\}$ is ∂-homogeneous if there is a positive integer d such that for all $\lambda \in \mathcal{U}$, $P(\lambda y_1, \lambda y_2) = \lambda^d P(y_1, y_2)$. Clearly, P must be a homogeneous polynomial. A subset of $\mathbb{P}^1(\mathcal{U})$ is *Kolchin closed* if it is the set of zeros of a finite set of ∂-homogeneous differential polynomials in $\mathcal{U}\{y_1, y_2\}$.

We refer to the equivalence classes $[u_1, u_2]$ as *points* on the projective line.

$\mathbb{P}^1(k)$ is covered by 2 affine open patches: $y_2 \neq 0$, and, $y_1 \neq 0$. We are interested only in the patch $O : y_2 \neq 0$. The point "at infinity" (not in O) is $[1, 0]$, which is in the other affine open patch. If $[u_1, u_2] \in O$, then, $[u_1, u_2] = [u, 1]$, where $u = \frac{u_1}{u_2}$. So, we identify O with $\mathbb{A}^1(k)$ by the map $[u_1, u_2] \longmapsto u$.

We identify $\mathbb{P}^1(k)$ with the extended k-line $k \cup \{\infty\} = O \cup \{[1, 0]\}$.

Set $k = \mathcal{U}$. $\mathcal{K} = \mathcal{U}^\partial$. The Kolchin closed set $\mathbb{P}^1(\mathcal{K}) = \{[c_1, c_2] : c_1, c_2 \in \mathcal{K}\}$ is defined by the differential polynomial

$$y_2 y_1' - y_2' y_1.$$

For the action of SL_2 on the set of Riccati equations, see "Integrability of the Riccati equation from a group theoretical viewpoint," José F. Cariñena, Artur Ramos, arXiv.org.math-ph19810005.

If V is the Riccati variety defined by $R(a_0, a_1, a_2)$, we embed it in the projective line as follows:

We homogenize $R(a_0, a_1, a_2)$: Set $y = \frac{y_1}{y_2}$.

$$\left(\frac{y_1}{y_2}\right)' = a_0 + a_1 \frac{y_1}{y_2} + a_2 \left(\frac{y_1}{y_2}\right)^2.$$

Multiply through by y_2^2. The homogenization is:

$$y_1'y_2 - y_1y_2' = a_0y_2^2 + a_1y_1y_2 + a_2y_1^2.$$

Lemma 4.1. *The affine Riccati variety equals its projective closure if and only if* $a_2 \neq 0$.

Proof. The point $[1, 0]$ at infinity satisfies the homogenization of the Riccati equation iff $a_2 = 0$. ☐

Let k be any field of characteristic zero. $PGL_2(k)$ is the group consisting of all *projective linear transformations*:

$$\tau(u) = \frac{\alpha u + \beta}{\gamma u + \delta} \qquad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(k).$$

Note that the transformation may send $u \in O$ "to ∞." This happens when $u = -\frac{\delta}{\gamma}$. τ is invertible, with inverse represented by M^{-1}.

The map $M \longmapsto \tau$, defined as above, is a surjective homomorphism from $GL_2(k)$ onto $PGL_2(k)$with kernel the center $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha \in \mathbb{G}_m(k)$. It defines on $PGL_2(k)$ the structure of algebraic group. If $k = \mathcal{U}$, it has defined on it the canonical structure of differential algebraic group. Both of these structures are inherited from the general linear group.

Suppose k is algebraically closed. Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(k)$. Then, $\frac{1}{\sqrt{\det M}}M = \begin{pmatrix} \frac{\alpha}{\sqrt{\det M}} & \frac{\beta}{\sqrt{\det M}} \\ \frac{\gamma}{\sqrt{\det M}} & \frac{\delta}{\sqrt{\det M}} \end{pmatrix} \in SL_2(k)$ and has the same image in $PGL_2(k)$. So, $PGL_2(k) = PSL_2(k)$.

Remark 4.2. 1. Given a projective linear transformation

$$\tau(u) = \frac{\alpha u + \beta}{\gamma u + \delta},$$

represented by the matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then, $\tau(\infty) = \infty$ if and only if $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$.

So, the projective linear transformations fixing ∞ are the affine transformations $\tau(u) = \varepsilon u + \eta$, $\varepsilon \in \mathbb{G}_m(k), \eta \in \mathbb{G}_a(k)$.

$J(u) = \frac{1}{u}$, represented by $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, interchanges 0 and ∞.

2. Let $k = \mathbb{C}$, the field of complex numbers. Projective linear transformations of $\mathbb{C} \cup \infty$ are often called Möbius transformations, and, the extended $\mathbb{C}$-line is called the extended complex plane.

   When k is algebraically closed, $PGL_2(k)$ is connected. However, this need not be the case if k is not algebraically closed.

   $PGL_2(\mathbb{R})$ is not connected. Multiply $M \in GL_2(\mathbb{R})$ by $\frac{1}{\sqrt{|\det M|}}$. Then, $\det\left(\frac{1}{\sqrt{|\det M|}} M\right) = \pm 1$, and, the images of M and $\frac{1}{\sqrt{|\det M|}} M$ in $PGL_2(\mathbb{R})$ are the same. We may assume, therefore, that $\det M = \pm 1$. The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has determinant -1, and, its square is the identity matrix. Therefore, if $\det M = -1$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M \right)$. Thus, $PGL_2(\mathbb{R})$ has two components: $PSL_2(\mathbb{R})$ and the projective transformations associated with matrices in $GL_2(\mathbb{R})$ with determinant -1.

   $PGL_2(\mathbb{R}) = PSL_2(\mathbb{R}) \cup J \cdot PSL_2(\mathbb{R})$, where $J(u) = \frac{1}{u}$, represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 4.3. *Given a triple* (u_1, u_2, u_3) *of distinct points in* $\mathbb{P}^1(k)$*, there is a unique projective linear transformation* λ *mapping* (u_1, u_2, u_3) *onto* $(0, 1, \infty)$*. It is given by the formula:*

$$\lambda(u) = \frac{(u - u_1)(u_2 - u_3)}{(u_1 - u_2)(u_3 - u)}$$

Note that the determinant of the representing matrix is

$$(u_1 - u_2)(u_2 - u_3)(u_3 - u_1).$$

So, it is invertible.

Definition 4.4. $\lambda(u)$ is called the *cross-ratio* (anharmonic ratio) of the quadruple (u, u_1, u_2, u_3).

Remark 4.5. The definition of cross-ratio is dependent on the order of the points u, u_1, u_2, u_3. There are 6 values of the cross-ratio under the 24 permutations of u, u_1, u_2, u_3 :

$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}.$$

Corollary 4.6. *Given two triples (u_1, u_2, u_3), (v_1, v_2, v_3) of distinct points in $\mathbb{P}^1(k)$, there is a unique projective linear transformation τ mapping (u_1, u_2, u_3) onto (v_1, v_2, v_3).*

Corollary 4.7. *Let (u, u_1, u_2, u_3) and (v, v_1, v_2, v_3)be quadruples of points in $\mathbb{P}^1(k)$ such that u_1, u_2, u_3 are distinct, as are v_1, v_2, v_3. Then, there is a projective linear transformation τ mapping (u, u_1, u_2, u_3) onto (v, v_1, v_2, v_3) if and only if their cross-ratios are equal.*

Proof. Suppose the transformation τ maps (u, u_1, u_2, u_3) onto (v, v_1, v_2, v_3). Let υ be the unique transformation mapping (v_1, v_2, v_3) onto$(0, 1, \infty)$. Then, $\upsilon(v) = \lambda(v)$. $\upsilon\tau(u, u_1, u_2, u_3) = (\lambda(u), 0, 1, \infty) = \upsilon(v, v_1, v_2, v_3) = (\lambda(v), 0, 1, \infty)$. So, their cross-ratios are equal.

Conversely, suppose $\lambda(u) = \lambda(v)$. Then, there exists υ, τ such that $\upsilon(u, u_1, u_2, u_3) = (\lambda, 0, 1, \infty) = \tau(v, v_1, v_2, v_3)$. Therefore, $\upsilon^{-1}\tau(u, u_1, u_2, u_3) = (v, v_1, v_2, v_3)$. □

Corollary 4.8. *Cross-ratio is an invariant of the projective linear group.*

A circle (which may be a straight line) in the extended complex plane $\mathbb{C} \cup \{\infty\}$ is uniquely determined by a triple (u_1, u_2, u_3) of points on the circle. The unique circle passing through $(0, 1, \infty)$ is $\mathbb{R} \cup \{\infty\}$.

All affine transformations $\tau(u) = \alpha u + \beta$, $\alpha \in \mathbb{G}_m(\mathbb{C})$, $\beta \in \mathbb{G}_a(\mathbb{C})$, transform circles into circles, since multiplication by α rotates and perhaps stretches or contracts the circle, and, translation moves the circle to another center. The transformation $u \longmapsto \frac{1}{u}$ reflects the circle in the real axis and perhaps stretches or contracts. So,

Lemma 4.9. *Every projective linear transformation maps a circle onto a circle.*

We now show that $PSL_2(\mathbb{C})$ acts transitively on the set of circles in the extended complex plane.

Proposition 4.10. *Let C and C' be circles in the extended complex plane. Then, there is a projective linear transformation τ mapping C onto C'.*

Proof. Let (u_1, u_2, u_3) be a triple of distinct points on C, and, (v_1, v_2, v_3) be a triple of distinct points on C'. Let τ be the unique transformation mapping (u_1, u_2, u_3) onto (v_1, v_2, v_3). $\tau(C)$ is a circle containing (v_1, v_2, v_3). Therefore, $\tau(C) = C'$. □

Proposition 4.11. *Let C be the circle in the extended complex plane through three distinct points u_1, u_2, u_3. Then,C is the set of all points $u \in \mathbb{P}^1(\mathbb{C})$ such that the cross-ratio*

$$\lambda(u) = \frac{(u-u_1)(u_2-u_3)}{(u_1-u_2)(u_3-u)}$$

of (u, u_1, u_2, u_3) is in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Proof. Let τ be the transformation that carries (u_1, u_2, u_3) to $(0, 1, \infty)$. Then, $\tau(C) = \mathbb{P}^1(\mathbb{R})$. □

Definition 4.12. Let k be a field, and let $S \subset \mathbb{P}^1(k)$. The set of all projective linear transformations τ mapping S into S (restricting to self-maps of S) is a subgroup of $PGL_2(k)$, called the *stabilizer* of S, and is denoted by $G(S)$.

Theorem 4.13. *Let C be a circle in $\mathbb{P}^1(\mathbb{C})$. The stabilizer $G(C)$ is conjugate to $PGL_2(\mathbb{R})$.*

Proof. There exists a projective linear transformation τ such that $\tau(C) = \mathbb{P}^1(\mathbb{R})$. The stabilizer $G(\mathbb{P}^1(\mathbb{R})) = PGL_2(\mathbb{R})$. Therefore, it is not hard to see that $\tau G(C)\tau^{-1} = G(\mathbb{P}^1(\mathbb{R}))$. □

Let τ be in $PGL_2(\mathbb{R})$. Since τ is a homeomorphism (of the Riemann sphere), it maps each of the disks bounded by C onto a disk bounded by $\mathbb{P}^1(\mathbb{R})$, namely, onto the upper half plane U or onto the lower half plane L. The stabilizer $G(U)$ of the upper half plane is $PSL_2(\mathbb{R})$, and, since the inversion map J maps U onto L, if the matrix representing τ has determinant -1, τ interchanges the upper and lower half planes. Let $S \in SL_2(\mathbb{R})$ represent a transformation $\sigma \in PSL_2(\mathbb{R})$, and, let $T \in SL_2(\mathbb{C})$ represent

$\tau \in PGL_2(\mathbb{C})$. Then $T^{-1}ST$ represents $\tau^{-1}\sigma\tau$. $\det(T^{-1}ST) = 1$. So, $T^{-1}ST \in SL_2(\mathbb{C})$. So, we have:

Theorem 4.14. *The stabilizer G in $SL_2(\mathbb{C})$ of a circle C in the extended complex plane under the action by $PSL_2(\mathbb{C})$ is conjugate in $SL_2(\mathbb{C})$ to $SL_2(\mathbb{R})$. G also stabilizes each disk bounded by the circle.*

4.1 The action of $SL_2(\mathcal{U})$ on the set of Riccati varieties

We identify the set of Riccati equations $R(a_0, a_1, a_2) : \ y' = a_0 + a_1 y + a_2 y^2$ with $\mathcal{U}^3 : R(a_0, a_1, a_2) \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$. $SL_2(\mathcal{U})$ acts on the set of Riccati equations as follows:

Let $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha\delta - \beta\gamma = 1$. Map $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$ to $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$, where

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ -2\alpha\gamma & \alpha\delta + \beta\gamma & -2\beta\delta \\ \gamma^2 & -\gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \alpha\beta' - \alpha'\beta \\ 2(\alpha'\delta - \beta'\gamma) \\ \gamma\delta' - \gamma'\delta \end{pmatrix}.$$

This action transforms $R(a_0, a_1, a_2)$ into $R(b_0, b_1, b_2)$. The inverse transformation is

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \delta^2 & \beta\delta & \beta^2 \\ 2\delta\gamma & \alpha\delta + \beta\gamma & 2\alpha\beta \\ \gamma^2 & \alpha\gamma & \alpha^2 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} \beta\delta' - \beta'\delta \\ 2(\alpha\delta' - \beta'\gamma) \\ \alpha\gamma' - \alpha'\gamma \end{pmatrix}.$$

Lemma 4.15. *Let V be the Riccati variety defined by $R(a_0, a_1, a_2)$. Let $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha\delta - \beta\gamma = 1.$. The projective linear transformation*

$$\tau(u) = \frac{\alpha u + \beta}{\gamma u + \delta}$$

transforms V into the Riccati variety W defined by $R(b_0, b_1, b_2)$, where $R(b_0, b_1, b_2)$ is the transform by Z of Riccati equation $R(a_0, a_1, a_2)$.

Proof. A tedious but straightforward computation shows that $\tau(V) \subset W$. τ^{-1} then maps W into V. So, τ restricts to an isomorphism of V onto W. □

Corollary 4.16. *Every projective linear transformation in $PSL_2(\mathcal{U})$ maps a Riccati variety onto a Riccati variety.*

Corollary 4.17. *Let (u_1, u_2, u_3) be a triple of distinct points in $\mathbb{P}^1(\mathcal{U})$. There is a unique Riccati variety containing u_1, u_2, u_3.*

Proof. Let $\tau^{-1}(u) = \frac{\alpha u+\beta}{\gamma u+\delta}$ be the unique transformation mapping (u_1, u_2, u_3) onto $(0, 1, \infty)$. Then, τ maps the Riccati variety $\mathbb{P}^1(\mathcal{K})$ onto a Riccati variety V containing u_1, u_2, u_3. If W is a Riccati variety containing the three points, then, $\tau(\mathbb{P}^1(\mathcal{K})) = W = V$. □

Corollary 4.18. *Let V be the Riccati variety in the extended $\mathcal{U}$-line through three distinct points u_1, u_2, u_3. Then, V is the set of all points $u \in \mathbb{P}^1(\mathcal{U})$ such that the cross-ratio*

$$\lambda(u) = \frac{(u-u_1)(u_2-u_3)}{(u_1-u_2)(u_3-u)}$$

of u, u_1, u_2, u_3 is in $\mathbb{P}^1(\mathcal{K})$.

Corollary 4.19. *Let V be the Riccati variety in the extended $\mathcal{U}$-line through three distinct points u_1, u_2, u_3. Then, V is the set of all points $u \in \mathbb{P}^1(\mathcal{U})$ such that*

$$u = \frac{c(u_1-u_2)+u_1(u_2-u_3)}{c(u_1-u_2)+(u_2-u_3)}, c \in \mathbb{P}^1(\mathcal{K}).$$

This formula, called a *superposition principle*, gives us the transformation τ. We have represented τ by a matrix in $GL_2(\mathcal{U})$. $\det\tau = (u_1-u_2)(u_2-u_3)(u_3-u_1)$. Notice that the elements of the Riccati variety depend *rationally* on one "arbitrary constant." In differential equations theory, the Riccati equation is a differential equation with no movable singularities (movable poles are allowed).

What do the stabilizers of Riccati varieties look like?

Theorem 4.20. *The stabilizer $G(V)$ of a Riccati variety in $PSL_2(\mathcal{U})$ is conjugate to $PSL_2(\mathcal{K})$.*

Proof. Let τ be a projective linear transformation mapping V onto $\mathbb{P}^1(\mathcal{K})$. The stabilizer of $\mathbb{P}^1(\mathcal{K})$ is $PSL_2(\mathcal{K})$. $\tau G(V)\tau^{-1} = PSL_2(\mathcal{K})$. □

4.2 The gauge action revisited

Recall that the gauge action of $SL_2(\mathfrak{U})$ on its Lie algebra $sl_2(\mathfrak{U})$ is defined by the formula:

$$A \longmapsto ZAZ^{-1} + \partial Z\ Z^{-1} \quad A \in sl_2(\mathfrak{U}), Z \in SL_2(\mathfrak{U}).$$

The inverse of the gauge transformation by Z is

$$A \longmapsto Z^{-1}AZ - Z^{-1}\partial Z.$$

We now describe this affine transformation of $sl_2(\mathfrak{U})$ explicitly. We choose the following basis of $sl_2(\mathfrak{U})$:

$$E_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Write $A = a_0E_0 + a_1E_1 + a_2E_2$ Then,

$$ZAZ^{-1} + \partial Z\ Z^{-1} = b_0E_0 + b_1E_1 + b_2E_2,$$

where

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ -2\alpha\gamma & \alpha\delta + \beta\gamma & -2\beta\delta \\ \gamma^2 & -\gamma\delta & \delta^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \alpha\beta' - \alpha'\beta \\ 2(\alpha'\delta - \beta'\gamma) \\ \gamma\delta' - \gamma'\delta \end{pmatrix}.$$

The inverse is

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \delta^2 & \beta\delta & \beta^2 \\ 2\delta\gamma & \alpha\delta + \beta\gamma & 2\alpha\beta \\ \gamma^2 & \alpha\gamma & \alpha^2 \end{pmatrix}\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} \beta\delta' - \beta'\delta \\ 2(\alpha\delta' - \beta'\gamma) \\ \alpha\gamma' - \alpha'\gamma \end{pmatrix}.$$

Note that the gauge action of $SL_2(\mathfrak{U})$ on $sl_2(\mathfrak{U})$ is the same as the action, described in section 4.1, of $SL_2(\mathfrak{U})$ on the set of Riccati equations.

Proposition 4.21. *Let G be a proper Zariski dense ∂-subgroup of $SL_2(\mathcal{U})$. There exists a projective Riccati variety V such that V is a homogeneous space for G under the action of G on $\mathbb{P}^1(\mathcal{U})$ by projective linear transformations. G is the isotropy group under the gauge action of the matrix A representing the defining equation of V.*

Proof. G is the isotropy group under the gauge action of a matrix $A = a_0E_0+a_1E_1+a_2E_3$. The matrix A represents a Riccati equation $R(a_0, a_1, a_2)$. We set V equal to the projective variety defined by the Riccati equation $R(a_0, a_1, a_2)$. G fixes $R(a_0, a_1, a_2)$, since it fixes A under the gauge action. Therefore, the image G' of G in the projective special linear group stabilizes V. Let τ be the unique transformation such that $\tau(V) = \mathbb{P}^1(\mathcal{K})$. Let u and v be in V. Let $c = \tau(u)$, and, $d = \tau(v)$. $PSL_2(\mathcal{K})$ acts transitively on $\mathbb{P}^1(\mathcal{K})$. So, there is a transformation $\sigma \in PSL_2(\mathcal{K})$, with $\sigma(c) = d$. Therefore, $\tau^{-1} \circ \sigma \circ \tau$ maps u to v, and, is in $G(V)$. □

So, we know that the stabilizers of Riccati varieties in the projective special linear group $PSL_2(\mathcal{U})$are precisely the images of simple ∂-groups that can be embedded in $SL_2(\mathcal{U})$ as proper subgroups. Note that if we know one element u_0 in a Riccati variety V then we can describe every element u in V as follows:

$$u = \frac{\alpha u_0 + \beta}{\gamma u_0 + \delta},$$

where $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in the isotropy group in $SL_2(\mathcal{U})$ of the matrix $A \in sl_2(\mathcal{U})$ under the gauge action.

For the saga of circles in the extended complex plane and their stabilizers, see Hans Schwerdtfeger, *Geometry of Complex Numbers*, 1962, and, Daniel Pedoe, *Circles: A Mathematical Point of View,* both republished by Dover Publications in 1979. An excellent discussion of Möbius transformations, cross-ratio, and, circles, from the point of view of complex function theory, is Gareth A. Jones, and David Singerman, *Complex Functions: An Algebraic and Geometric Viewpoint, 1987.* For a contrasting discussion of Riccati varieties from the viewpoint of differential equations theory, see Earl D. Rainville, *Intermediate Differential Equations*, 1943.