A Brief Introduction to Schemes

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Throughout these notes A denotes a commutative ring with $1 \neq 0$. More generally, all rings are assumed commutative.

1 The Prime Spectrum of a Ring

The *prime spectrum* A, or simply the *spectrum* of A, is defined by

$$\mathrm{Spec\,A} = \{\mathfrak{p} \subset \mathrm{A} | \mathfrak{p} \text{ is a prime ideal}\}.$$

For any $S \subset A$ define

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec\,A} | \mathrm{S} \subset \mathfrak{p}\} \subset \mathrm{Spec\,A}.$$

Proposition 1.

1. $\emptyset = V(A)$ *and* $\mathrm{Spec\,A} = \mathrm{V}(0)$.

2. *If* S *and* T *are any two subsets of* A *then* $V(ST) = V(S) \cup V(T)$.

3. *If* $\{S_\alpha\}$ *is a set of subsets of* A *then* $\cap_\alpha V(S_\alpha) = V(\cup_\alpha S_\alpha)$.

Proof. 1. This is an easy consequence of the definition.

2. Let $\mathfrak{p} \in V(ST)$, i.e. $ST \subset \mathfrak{p}$. Suppose there exists $s \in S$ such that $s \notin \mathfrak{p}$. Then $st \in \mathfrak{p}$ for all $t \in T$ implies $T \subset \mathfrak{p}$ since $\mathfrak{p}$ is prime. Therefore $\mathfrak{p} \in V(T) \subset\in V(S) \cup V(T)$. To prove the opposite inclusion let $\mathfrak{p} \in V(S) \cup V(T)$. Without loss of generality we can assume $\mathfrak{p} \in V(S)$. So $st \in \mathfrak{p}$ for all $s \in S,\ t \in T$ therefore $\mathfrak{p} \in V(ST)$.

3. $\cap_\alpha V(S_\alpha) = \{\mathfrak{p} \in \mathrm{Spec\,A} | \mathrm{S}_\alpha \subset \mathfrak{p} \text{ for all } \alpha\} = \{\mathfrak{p} \in \mathrm{Spec\,A} | \cup_\alpha \mathrm{S}_\alpha \subset \mathfrak{p}\} = \mathrm{V}(\cup_\alpha \mathrm{S}_\alpha)$. □

As a consequence we can define the *Zariski topology* on $\operatorname{Spec} A$ to be topology with closed sets of the form $V(S)$ for $S \subset A$. The complementary open sets $V(S)^c$ are denoted $D(S)$. The open sets of the form $D(f) = (V(\{f\}))^c = \{\mathfrak{p} \in \operatorname{Spec} A | f \notin \mathfrak{p}\}$, $f \in A$, are the *basic open sets*; they form a base for the topology. One has $D(f) \cap D(g) = D(fg)$. The Zariski topology is henceforth assumed on $\operatorname{Spec} A$.

When $E \subset \operatorname{Spec} A$ the relative topology on E is assumed. Example: $E = \{\mathfrak{m} \in \operatorname{Spec} A | \mathfrak{m} \text{ is maximal}\}$ is the *maximal spectrum* of A and is denoted MaxSpec A; in this case one again refers to the *Zariski topology.* When A is a differential ring the subset $E = \{\mathfrak{p} \in \operatorname{Spec} A | \mathfrak{p} \text{ is a prime differential ideal}\}$ is the *differential spectrum of* A; here one refers to the *Kolchin topology* and writes DiffSpecA.

Proposition 2.

1. *If $S, T \subset A$ satisfy $T \subset S$ then $V(S) \subset V(T)$.*

2. *If $S \subset A$ then $V(S) = V(\mathfrak{i})$, where $\mathfrak{i}$ is the ideal generated by S.*

Proof. 1. This is an easy consequence of the definition.

2. $V(\mathfrak{i}) \subset V(S)$ from 1. Let $\mathfrak{p} \in V(S)$. Let $a \in \mathfrak{i}$ so $a = a_1 s_1 + \cdots + a_n s_n$ for some $a_j \in A$ and $s_j \in S$. $s_j \in \mathfrak{p}$ for all j hence $a \in \mathfrak{p}$. Therefore $i \subset \mathfrak{p}$ and $\mathfrak{p} \in V(i)$. □

Given a closed set V of $\operatorname{Spec} A$ define

$$I(V) = \{f \in A | f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in \text{V}\}.$$

Recall that the *radical of an ideal*, $\mathfrak{i} \subset A$, is

$$\sqrt{\mathfrak{i}} = \{f \in A | f^n \in \mathfrak{i} \text{ for some n} \in \mathbb{N}\}.$$

If $\sqrt{\mathfrak{i}} = \mathfrak{i}$ then $\mathfrak{i}$ is a *radical ideal.*

Proposition 3.

1. *If $V \subset \operatorname{Spec} A$ is closed then $I(V) = \cap_{\mathfrak{p} \in V} \mathfrak{p}$ and therefore $I(V)$ is an ideal.*

2. *If $\mathfrak{i}$ and $\mathfrak{j}$ are two ideals in A then $V(\mathfrak{i}) \subset V(\mathfrak{j})$ if and only if $\sqrt{\mathfrak{i}} \supset \sqrt{\mathfrak{j}}$.*

3. *There is a bijection between radical ideals in A and closed sets in* $\operatorname{Spec} A$.

$I(V)$ is the *defining ideal* of V.

Proof. 1. This is an easy consequence of the definition.

2.

$$I(V(\mathfrak{i})) = \cap_{\mathfrak{p}\in V(\mathfrak{i})}\mathfrak{p} = \sqrt{\mathfrak{i}}$$

and similarly for $\mathfrak{j}$. The result follows.

3. If $\mathfrak{i}$ is a radical ideal in A then $V(\mathfrak{i})$ is a closed set in Spec A. Therefore[1]

$$I(V(\mathfrak{i})) = \cap_{\mathfrak{p}\in V(\mathfrak{i})}\mathfrak{p} = \sqrt{\mathfrak{i}} = \mathfrak{i}$$

and the result follows. □

Proposition 4. $X = \text{Spec A}$ *is quasi-compact* [2].

Proof. Let $\{U_\alpha\}$ be an open cover of X. The basic open sets for a base for the topology so we can refine the open cover to $\{D(f_\beta)\}$. $X = \cup D(f_\beta)$ is equivalent to $\cap V(\{f_\beta\}) = \emptyset$ which implies that the f_β generate the ring A. In other words there exist $f_1, \ldots, f_n$ and $a_1, \ldots, a_n$ such that $1 = \sum_1^n a_i f_i$. Therefore

$$X = \cup_1^n D(f_i)$$

□

2 Sheaves

Let X be a topological space and $\mathcal{C}$ the category in which the objects are open sets in X and the morphisms are inclusion mappings. A *pre-sheaf of rings* is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$, the category of rings. $\mathcal{D}$ can be replaced with certain other categories (sets, abelian groups, modules, ...) and one obtains a pre-sheaf in that category. To be more explicit, the conditions required for $\mathfrak{F}$ to be a pre-sheaf of rings on a topological space are:

1. if $U \subset X$ is open then $\mathfrak{F}(U) = R$; where R is a ring;

2. if $U, V \subset X$ are open such that $V \subset U$ then there exists a ring homomorphism $\rho_V^U : \mathfrak{F}(U) \to \mathfrak{F}(V)$; and

[1] See Atiyah-Macdonald [A-M] p. 9

[2] A topological space is *quasi-compact* if every open cover has a finite sub-cover. In algebraic geometry the term compact is commonly reserved for Hausdorff topological spaces.

3. if $U, V, W \subset X$ are open such that $W \subset V \subset U$ then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

The induced ring homomorphisms are usually called *restriction maps* and the elements of $\mathfrak{F}(U)$ are called the *sections* of U.

A *sheaf of rings* $\mathfrak{F}$ is a pre-sheaf of rings that satisfies the following additional properties.

1. Let $U \subset X$ be open and $\{V_i\}_I$ be a family of open sets such that $U = \cup V_i$. If $r \in \mathfrak{F}(U)$ then $r = 0$ if $\rho^U_{V_i}(r) = 0$ for all $i \in I$.

2. Let $U \subset X$ be open and $\{V_i\}_I$ be a family of open sets such that $U = \cup V_i$. Suppose for every $i, j \in I$ there exist $r_i \in \mathfrak{F}(V_i)$ and $r_j \in \mathfrak{F}(V_j)$ such that $\rho^{V_i}_{V_i \cap V_j}(r_i) = \rho^{V_j}_{V_i \cap V_j}(r_j)$ then there exists a unique $r \in \mathfrak{F}(U)$ such that $\rho^U_{V_i}(r) = r_i$ for all $i \in I$.

One defines sheaves with values in other categories by replacing the category of rings with the appropriate category.

For any topological space X we can form a sheaf of rings by letting $\mathfrak{F}(U)$ be the ring of continuous real valued functions on U and letting the restriction maps be the natural restrictions of these functions. It is easy to verify that the properties of a sheaf are satisfied. In a similar manner we can form the sheaf of holomorphic functions when X is a complex manifold.

We give an example of a pre-sheaf of abelian groups that is not a sheaf. Let X be a topological space and G an abelian group. Define a pre-sheaf, $\mathfrak{F}$, on X by $\mathfrak{F}(X) = G$ and $\mathfrak{F}(U) = 0$, the trivial zero group, for all open $U \subset X$. The restriction maps take any element in $\mathfrak{F}(X) = G$ to the zero element and otherwise map 0 to 0. It follows easily that $\mathfrak{F}$ is a pre-sheaf but it fails to satisfy the two additional required of a sheaf.

If X and Y are topological spaces, $f : X \to Y$ is continuous and $\mathfrak{F}$ is a sheaf on X then the *direct image sheaf* $f_*\mathfrak{F}$ on Y is defined by $f_*\mathfrak{F}(U) = \mathfrak{F}(f^{-1}(U))$ for every open $U \subset Y$. It is left to the reader to show that this is a sheaf on Y.

Let $\mathfrak{F}$ be a sheaf of rings on X and $p \in X$. The *stalk* $\mathfrak{F}_{\mathfrak{p}}$ of $\mathfrak{F}$ at p is defined as the direct limit of $\mathfrak{F}(U)$ for all open $U \subset X$ such that $p \in X$. Recall that the direct limit is the set of equivalence classes in $\amalg_{V \subset U} \mathfrak{F}(V)$ where $s \sim t$ if $\rho^{V_S}_{V_s \cap V_T}(s) = \rho^{V_t}_{V_s \cap V_T}(t)$. The elements of $\mathfrak{F}_p$ are the germs of sections at p. A good example is the germs of continuous (holomorphic) functions at $x \in X$.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two sheaves of rings on X. A morphism ϕ of sheaves is a family of morphisms $\phi_U : \mathfrak{F}(U) \to \mathfrak{G}(U)$ indexed by the open sets $U \subset X$

such that if $V \subset U$ is open then $\phi_V \circ \psi_V^U(x) = \rho_V^U \circ \phi_U(x)$ for all $x \in U$, where the ρ and ψ are restriction maps for $\mathfrak{F}$ and $\mathfrak{G}$ respectively. In other words the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{F}(U) & \xrightarrow{\phi_U} & \mathfrak{G}(U) \\ \rho_V^U \downarrow & & \downarrow \psi_V^U \\ \mathfrak{F}(V) & \xrightarrow{\phi_V} & \mathfrak{G}(V) \end{array}$$

Note that we have defined morphisms between sheaves over the same topological space X. It is more complicated to define morphisms between sheaves over different topological spaces. This will be more evident during the discussion of schemes.

3 The Sheaf of Regular Functions

Let A be a ring and let $X = \operatorname{Spec} \mathrm{A}$ endowed with the Zariski topology. We shall build a sheaf of rings $\mathcal{O}$ on X. For every $\mathfrak{p} \in X$ let[3] $A_\mathfrak{p}$ be the corresponding local ring and define $E = \amalg_{\mathfrak{p} \in X} A_\mathfrak{p}$. We can define maps $s : X \to E$ where $s(\mathfrak{p}) \in A_\mathfrak{p}$. Let $\mathcal{O}(U)$ be the set of all s such that for all $\mathfrak{p} \in U$ there exists an open neighborhood $V \subset U$ of $\mathfrak{p}$ such that $s(\mathfrak{q}) = \frac{a}{b}$ where $a \in A$ and $b \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$. Note that a and b as elements of A are fixed and $s(\mathfrak{q}) = \frac{a}{b} \in A_\mathfrak{q}$ for all $\mathfrak{q} \in V$. These s are called the *regular functions* on U.

It is clear that $\mathcal{O}(U)$ is a ring with $(s+t)(\mathfrak{p}) = s(\mathfrak{p}) + t(\mathfrak{p})$ and $(st)(\mathfrak{p}) = s(\mathfrak{p})t(\mathfrak{p})$.

Proposition 5. *$\mathcal{O}$ is a sheaf of rings on X.*

Proof. Let $U \subset X$ be open. In order to verify that $\mathcal{O}$ is a sheaf we shall start by verifying that it is a pre-sheaf. Let $s \in \mathcal{O}(U)$ and $V \subset U$ be open. If $\mathfrak{p} \in V$ then $\mathfrak{p} \in U$ so s is defined at $\mathfrak{p}$ hence there exists an open neighborhood W of $\mathfrak{p}$ such that $s(\mathfrak{q}) = \frac{a}{f}$ where $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in W$. Note that $W \cap V$ is an open neighborhood of $\mathfrak{p}$ contained in V which demonstrates that s is a regular function on V. This defines the restriction map $\rho_V^U(s) = s|_V$. It is easy to verify the remaining properties of a pre-sheaf.

[3] $A_\mathfrak{p}$ denotes the localization of A at $\mathfrak{p}$, see Atiyah-Macdonald [A-M] p. 38.

Let $\{V_i\}$ be an open cover of U. Suppose there exist regular functions $s_i \in \mathcal{O}(V_i)$ such that $\rho^{V_i}_{V_i \cap V_j}(s_i) = \rho^{V_j}_{V_i \cap V_J}(s_j)$ for all i and j. Let $\mathfrak{p} \in U$ so $\mathfrak{p} \in V_i$ for some i, define $s(\mathfrak{p}) = s_i(\mathfrak{p})$. s is well-defined due to the assumption on the s_i that they agree on overlaps. s is a regular function because if $\mathfrak{p} \in U$ then $\mathfrak{p} \in V_i$ for some i and there is an open neighborhood of $\mathfrak{p}$ in V_i such that s_i satisfies the definition of a regular function. But this neighborhood is also a subset of U, consequently $s \in \mathcal{O}(U)$. Uniqueness follows. $\square$

$\mathcal{O}$ is the *sheaf of regular functions* of Spec A or the *structure sheaf* of Spec A.

Proposition 6. *Let A be a ring and $\mathcal{O}$ the sheaf of regular functions on $X = \text{Spec A}$ then the following are true.*

1. *$A_{\mathfrak{p}}$ is isomorphic to $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$.*

2. *If A is reduced*[4] *then A is isomorphic to the global sections on X, i.e. A is isomorphic to $\mathcal{O}(D(1))$.*

3. *$\mathcal{O}(D(f))$ is isomorphic to*[5] *A_f for all $f \in A$.*

Note that 2 is a special case of 3. As the proof of 3 is tedious and 2 is interesting in its own right we only give the proof of 2.

Proof. 1. Let $\mathfrak{p} \in X$ and define $\phi : \mathcal{O}_{\mathfrak{p}} \to A_{\mathfrak{p}}$ so that $\phi(s) = s(\mathfrak{p})$. This is well-defined because if $s \sim t$ then $s(\mathfrak{p}) = t(\mathfrak{p})$. Let s, t be two representatives and suppose $\phi(s) = s(\mathfrak{p}) = t(\mathfrak{p}) = \phi(t)$. Let $s(\mathfrak{p}) = \frac{a}{f}$ and $t(\mathfrak{p}) = \frac{b}{g}$ where $a, b \in A$ and $f, g \notin \mathfrak{p}$. $s(\mathfrak{p}) = t(\mathfrak{p})$ implies that there exists $c \notin \mathfrak{p}$ such that $c(ag - bf) = 0$. Let $V = D(f) \cap D(g) \cap D(c)$ and note that $\mathfrak{p} \in V$. In addition $\frac{a}{f} = \frac{b}{g}$ in $A_{\mathfrak{q}}$ for every $\mathfrak{q} \in V$. Then s and t agree on V, an open neighborhood of $\mathfrak{p}$, hence $s \sim t$ in $\mathcal{O}_{\mathfrak{p}}$. So ϕ is injective. Let $\frac{a}{f} \in A_{\mathfrak{p}}$ so $a \in A$ and $f \notin \mathfrak{p}$. Note that $D(f)$ is an open set containing $\mathfrak{p}$ and define $s(\mathfrak{q}) = \frac{a}{f}$ for all $\mathfrak{q} \in D(f)$ therefore $s \in \mathcal{O}(D(f))$. Hence $\phi(s) = \frac{a}{f}$ and we have that ϕ is surjective. Therefore $\phi : \mathcal{O}_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is an isomorphism.

2. First note that $D(1) = X$. Define $\phi : A \to \mathcal{O}(D(1))$ by $\phi(a) = s \in \mathcal{O}(D(1))$, where $s(\mathfrak{p}) = \frac{a}{1} \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$. Let $a, b \in A$ and suppose that $\phi(a) = \phi(b) = s \in \mathcal{O}(D(1))$. Therefore $s(\mathfrak{p}) = \frac{a}{1} = \frac{b}{1} \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$

[4] A ring is reduced if it has no non-zero nilpotent elements

[5] A_f denotes the ring of fractions $S^{-1}A$ where $S = \{f, f^2, f^3, \ldots\}$, see Atiyah-Macdonald [A-M] p. 38

hence there exists $c \in A$ such that $c(a-b)=0$ and $c \notin \mathfrak{p}$ for all $\mathfrak{p} \in X$. The second condition implies that c is a unit therefore $a=b$. Hence ϕ is injective.

Let $s \in \mathcal{O}(X)$. We can create an open cover $\{U_\alpha\}$ of X such that $s(\mathfrak{p}) = \frac{a_\alpha}{f_\alpha}$ for all $\mathfrak{p} \in U_\alpha$, where $a_\alpha \in A$ and $f_\alpha \notin \mathfrak{q}$ for all $\mathfrak{q} \in U_\alpha$. The sets $D(f)$ form a base for the topology so we can cover each U_α with sets of the form $D(f_i)$ with $f_i \in A$. Note that $D(f_i) \subset D(f_\alpha)$ so $\sqrt{(f_i)} \subset \sqrt{(f_\alpha)}$, which implies that $f_i^n = cf_\alpha$ for some $c \in A$. Now we have $\frac{a_\alpha}{f_\alpha} = \frac{ca_\alpha}{f_i^n}$. $D(f_i) = D(f_i^n)$, which implies that after reindexing we can cover X with $D(f_i)$ such that $s(\mathfrak{p}) = \frac{a_i}{f_i}$ for all i. Since X is quasi-compact, we can assume the index set is finite.

$D(f_i) \cap D(f_j) = D(f_i f_j) = \emptyset$ when $f_i f_j \in \mathfrak{p}$ for all $\mathfrak{p} \in X$ which would imply that $f_i f_j$ is nilpotent[6]. A is reduced so $D(f_i) \cap D(f_j) \neq \emptyset$ for all i and j. Then for all i and j $s(\mathfrak{p}) = \frac{a_i}{f_i} = \frac{a_j}{f_j} = \frac{a}{f}$ for all $\mathfrak{p} \in D(f_i f_j)$. This implies that $f \notin \mathfrak{p}$ for all $\mathfrak{p} \in X$; therefore f is a unit. If $b = f * \frac{a}{f}$ then $s(\mathfrak{p}) = b$ for all $\mathfrak{p} \in X$ and we have that ϕ is surjective.

3. The general case of 2. is left to the reader (See Hartshorne [H] p. 71). □

This construction of the sheaf of regular functions follows Hartshorne. There is an equivalent construction in Eisenbud and Harris that first defines $\mathcal{O}(D(f)) = A_f$ for all $f \in A$ and then uses some sheaf theory to show that this correctly defines a sheaf on all of X. Both approaches have their merits. Hartshorne's construction uses less sheaf theory but requires the previous proposition, where as that result follows directly from the other construction. It is good to understand both.

One thing to note is that if $D(f) = D(g)$ for some$f, g \in A$ then $A_f = A_g$. If $\mathbb{Z} = A$ then this is the same as saying if two integers x and y have the same elements in their prime decomposition then the localizations are equal, i.e. $\mathbb{Z}_x = \mathbb{Z}_y$. This is true because there exists $c_1, c_2 \in \mathbb{Z}$ and $n, m \in N$ such that $x^n = c_1 y$ and $y^m = c_2 x$. For a general ring A this comes from the fact that if $D(f) = D(g)$ then $\sqrt{(f)} = \sqrt{(g)}$ which gives $c_1, c_2 \in A$ and $n, m \in \mathbb{N}$ such that $f^n = c_1 g$ and $g^m = c_2 f$.

[6]The *nilradical* of A, usually denoted as $\mathfrak{R}$, is the intersection of all prime ideals which is equal to the set of nilpotent elements of A, see Atiyah-Macdonald [A-M] p. 5

4 DiffSpec, a Digression

For this section let A be a differential ring. If $S \subset A$ then $[S]$ is the differential ideal generated by S, i.e. the smallest differential ideal containing S. If $\mathfrak{i}$ is a differential ideal then $\{i\}$ denotes the smallest radical differential ideal containing $\mathfrak{i}$. We have already mentioned $D = \text{DiffSpecA}$, the set of prime differential ideals, now define $D_\partial(f) = \{\mathfrak{p} \in D | f \notin \mathfrak{p}\}$. It is possible to construct a sheaf of regular functions on D in a manner completely similar to the previous section, although we will not go into the account here. For a complete construction see Kovacic [K].

However we would like to point out a challenge that occurs when working with a differential ring. Suppose $f, g \in A$ such that $D_\partial(f) = \{\mathfrak{p} \in D | f \notin \mathfrak{p}\} = \{\mathfrak{p} \in D | g \notin \mathfrak{p}\} = D_\partial(g)$. It follows that $\{f\} = \{g\}$ but now we are working with radical *differential* ideals. In contrast to the SpecA case we now get that $f^n = c \sum c_n g^{(n)}$, a multiple of a finite sum of g and the derivatives of g. This prevents us from obtaining $A_f = A_g$ in general as is the case with SpecA.

5 Schemes

A *ringed space* is a pair $(X, \mathcal{O}_X)$ such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A *locally ringed space* is a ringed space in which the stalks are local rings. Note that if $X = \text{Spec A}$ for some ring A and $\mathcal{O}_X$ is the sheaf of regular functions then $(X, \mathcal{O}_X)$ is a locally ringed space.

A morphism of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $f : X \to Y$ is continuous and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves. A morphism of locally ringed spaces is a morphism of ringed spaces such that the morphism of sheaves $f^\sharp$ preserves the local ring structure of the stalks. In other words if $p \in Y$ then $f^\sharp$ induces a local ring homomorphism $f^\sharp_p : (\mathcal{O}_Y)_p \to (f_*\mathcal{O}_X)_p$ such that $(f^\sharp_p)^{-1}(\mathfrak{m}_2) = \mathfrak{m}_1$ where $\mathfrak{m}_1$ is the maximal ideal in $\mathcal{O}_{Yp}$ and $\mathfrak{m}_2$ is the maximal ideal in $f_*\mathcal{O}_{Xp}$. The introduction of $f_*\mathcal{O}_X$ is required in this definition of morphisms because the two sheaves are over different topological spaces.

An *affine scheme* is a locally ringed space that is isomorphic as a locally ringed space to $(X, \mathcal{O}_X)$, where $X = \text{Spec A}$ and $\mathcal{O}_X$ is the associated structure sheaf. Note that $(X, \mathcal{O}_X)$ is trivially an affine scheme. A *scheme* is a locally ringed space, $(X, \mathcal{O}_X)$, that is covered by affine schemes. In other words, for every $x \in X$ there exists an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is isomorphic as a locally ringed space to an affine scheme.

Proposition 7. *If A and B are rings and $\phi : A \to B$ is a homomorphism then there is a induced morphism of locally ringed spaces*

$$(f, f^\sharp) : (\text{Spec B}, \mathcal{O}_\text{B}) \to (\text{Spec A}, \mathcal{O}_\text{A}).$$

Proof. Let $X^A = \text{Spec A}$ and $X^B = \text{Spec B}$. ϕ induces a map $f : X^B \to X^A$ with $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in X^B$; we need to show that f is continuous. Let $V \subset X^A$ be closed. $V = \{\mathfrak{q} \in X^A | \mathfrak{i} \subset \mathfrak{q}\}$ for some ideal $\mathfrak{i} \subset A$. $f^{-1}(V) = \{\mathfrak{p} \in X^B | \phi(i) \subset \mathfrak{p}\}$ which is closed. Hence f is continuous.

For any open $U \subset X^A$ define a ring homomorphism

$$f^\sharp_U : \mathcal{O}_A(U) \to f_*\mathcal{O}_B(U).$$

To this end choose $\mathfrak{p} \in f^{-1}(U)$ and let $\mathfrak{q} = f(\mathfrak{p})$. Then for any $s \in \mathcal{O}_A(U)$ there exists an open neighborhood $V \subset U$ of $\mathfrak{q}$ such that $s(\mathfrak{q}') = \frac{a}{b}$ for all $\mathfrak{q}' \in V$ with $b \notin \mathfrak{q}'$. Note that $\phi(b) \notin \mathfrak{p}'$ for all $\mathfrak{p}' \in f^{-1}(V)$. Since $\mathfrak{p} \in f^{-1}(V)$ we can define $f^\sharp_U(s) = t$ by $t(\mathfrak{p}) = \phi(s(f(\mathfrak{p}))) = \phi(s(\mathfrak{q})) = \frac{\phi(a)}{\phi(b)}$. The verifications that $f^\sharp$ is a morphism of sheaves and preserves the local ring structure of the stalks is left to the reader. $\square$

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